Constantes de Siegel-Veech et volumes de strates d'espaces de modules de différentielles quadratiques

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Résumé

Nous étudions les constantes de Siegel–Veech pour les surfaces plates et leurs liens avec les volumes de strates d'espaces de modules de différentielles quadratiques. Les constantes de Siegel–Veech donnent l'asymptotique du nombre de géodésiques périodiques dans les surfaces plates. Pour certaines surfaces plates, de telles géodésiques correspondent aux trajectoires périodiques dans les billards rationnels correspondants. Les constantes de Siegel–Veech sont fortement reliées à la dynamique du flot géodésique dans les espaces de modules correspondants, par la formule d'Eskin–Kontsevich–Zorich exprimant la somme des exposants de Lyapunov du fibré de Hodge le long du flot de Teichmüller en fonction de la constante de Siegel–Veech pour la strate considérée et d'un terme combinatoire explicite. Cette dynamique est liée à la dynamique du flot linéaire dans la surface plate de départ par un procédé de renormalisation. En utilisant certaines propriétés de cette dynamique nous montrons un critère qui détermine quand une courbe complexe plongée dans l'espace de module des surfaces de Riemann munie d'un sous-fibré en droites du fibré de Hodge est une courbe de Teichmüller.

Nous étudions certains rapports de constantes de Siegel–Veech et en déduisons des informations géométriques sur les régions périodiques dans les surfaces plates. Les liens entre les constantes de Siegel–Veech et les volumes d'espaces de modules ont été étudiés complètement dans le cas abélien par Eskin, Masur et Zorich, et dans le cas quadratique en genre zéro par Athreya, Eskin et Zorich. Nous généralisons ces résultats au cas quadratique en genre supérieur, en utilisant la description des configurations de liens selles produite par Masur et Zorich. Nous calculons de façon explicite certains volumes de strates de petite dimension.

Abstract

We study Siegel–Veech constants for flat surfaces and their links with the volumes of some strata of moduli spaces of quadratic differentials. Siegel–Veech constants give the asymptotics of the number of periodic geodesics in flat surfaces. For certain flat surfaces such geodesics correspond to periodic trajectories in related rational billiards. Siegel–Veech constants are strongly linked to the dynamics of the geodesic flow in related moduli spaces by the formula of Eskin–Kontsevich–Zorich, giving the sum of the Lyapunov exponents for the Hodge bundle along the Teichmüller geodesic flow in terms of the Siegel–Veech constant for the corresponding stratum and an explicit combinatorial expression. This dynamics is related to the dynamics of the linear flow in the original flat surface by a renormalization process. Using some properties of this dynamics we prove a criterion to detect whether a complex curve, embedded in the moduli space of Riemann surfaces and endowed with a line subbundle of the Hodge bundle, is a Teichmüller curve.

We study ratios of Siegel–Veech constants and deduce geometric informations about the periodic regions in flat surfaces. The links between Siegel–Veech constants and volumes of moduli spaces were completely studied by Eskin, Masur and Zorich in the Abelian case, and by Athreya, Eskin and Zorich in the quadratic case in genus zero. We generalize their results to the quadratic case in higher genus, using the description of configurations of saddle-connections performed by Masur and Zorich. We provide explicit computations of volumes of some strata of low dimension.

Remerciements

En premier lieu je voudrais remercier mon directeur Anton Zorich, qui m'a expliqué ce sujet avec passion, et m'a toujours aidée, soutenue, encouragée au cours de cette thèse. Il l'a encadrée avec sa générosité, sa gentillesse et son humilité naturelles qui font de lui un directeur inestimable.

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Au cours de ces trois années j'ai effectué mes missions d'enseignement à l'ENS et l'ENSAI, j'aimerais remercier tous les professeurs avec qui j'ai travaillé, les responsables d'enseignement Arnaud Debussche et Nicolas Souêtre, et mes élèves, pour avoir rendu ces missions particulièrement agréables. J'ai également particulièrement apprécié participer à des actions de diffusion des mathématiques, organisées en grande partie par Rozenn Texier-Picard. Je remercie tous les protagonistes de ces projets pour les échanges enrichissants qu'on a pu avoir. Ces deux aspects autres que la recherche m'ont apporté un certain équilibre pendant ma thèse.

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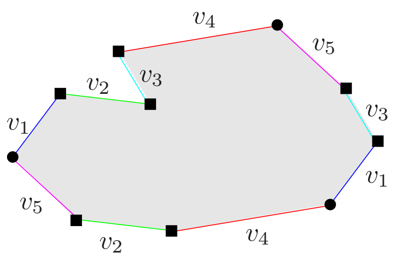
1 Présentation du sujet

1.1 Surfaces plates

Par *surfaces plates* nous désignerons les surfaces de translation et de demi-translations que nous définissons maintenant.

Considérons les polygones de la figure suivante. Si nous identifions les côtés parallèles et égaux du polygone de gauche par translation, nous obtenons une surface fermée qui hérite de la métrique du plan sauf aux points représentés par les sommets du polygone : certains de ces points sont identifiés et forment un angle 2π : autour de ce point la métrique est encore plate, les autres s'identifient pour former un angle 6π : en ce point, la métrique a une singularité conique. Par l'identité de Gauss-Bonnet nous déduisons donc que cette surface est de genre 2.

Si nous considérons surface obtenue en identifiant les côtés parallèles et égaux su polygone de droite par translation ou par translation et demi-tour, selon l'orientation suggérée, nous obtenons une surface fermée munie d'une métrique plate à singularités coniques avec deux singularités d'angle 5π et deux d'angle π. Par l'identité de Gauss-Bonnet nous déduisons donc que cette surface est de genre 2.



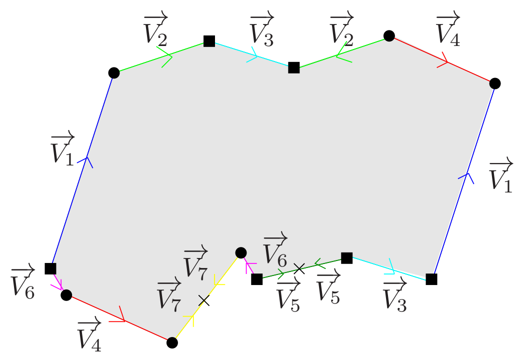


FIGURE 1 – Surfaces de translation et de demi-translation

Ce sont des exemples de surfaces de translation et de demi-translation. Pour formaliser ces notions, nous introduisons les définitions suivantes.

Définition 1. Nous appellerons *surface de translation* une surface de Riemann S munie d'une 1-forme holomorphe ω appelée *différentielle abélienne.*

Cette appellation est justifiée par l'équivalence à la définition suivante :

Définition 2. Une surface topologique compacte orientable S de genre g munie d'un sous-ensemble fini de points $\Sigma = \{P_1; \dots, P_n\}$ et d'une partition $d = (d_1, d_2, \dots, d_n)$ de $2g-2$ possède une structure de surface de translation si il existe un atlas maximal ϕ de cartes allant de $S \setminus \Sigma$ dans des ouverts de $\mathbb{C} \simeq \mathbb{R}^2$ tel que les changements de cartes soient des translations, et tel que pour tout i dans $\{1, \dots, n\}$ il existe un voisinage U_i de P_i, un voisinage V_i de 0 dans $\mathbb{R}^2$ et un revêtement ramifié $p = (U_i, P_i) \to (V_i, 0)$ de degré $d_i + 1$ tel que chaque restriction injective de p est une carte de ϕ.

Si (S,ω) est une surface de translation alors en notant Σ l'ensemble des zéros de ω et $(d_1,\dots,d_n)$ leur ordres correspondants, sa structure de surface de translation est donnée par les cartes locales ϕ_P au voisinage $U(P)$ de P point régulier de ω, définies par :

$$\begin{array}{cccc} \phi_P : & U(P) & \to & \mathbb{C} \\ & Q & \mapsto & \int_P^Q \omega. \end{array}$$

Réciproquement une structure de surface de translation sur (S,Σ,d) définit une structure de surface de Riemann sur S et une différentielle abélienne ω par le tiré en arrière de la forme $\mathrm{d}z$ sur $\mathbb{C}$ par les cartes de ϕ.

En plus de cette structure, une surface de translation (S,ω) hérite naturellement

- d'une métrique plate définie par ω, les zéros de ω de degré d correspondant à des singularités d'angle $2(d+1)\pi$.
- d'une orientation privilégiée donnée par l'orientation de $\mathbb{C}$,
- d'une forme d'aire donnée par $\frac{i}{2}\omega\wedge\overline{\omega}$,
- d'un champ de vecteurs parallèles privilégié (par convention le champ de vecteurs verticaux allant vers le nord)

Définition 3. Nous appelons *surface de demi-translation* une surface de Riemann S munie d'une *différentielle quadratique méromorphe* q, c'est-à-dire une forme donnée en coordonnées locales par $f(z)(dz)^2$ où f est une fonction méromorphe à pôles au plus simples.

Si q est globalement le carré d'une 1-forme holomorphe on retrouve le cas précédent.

Cette appellation est justifiée par l'équivalence à la définition suivante :

Définition 4. Une surface topologique compacte orientable S de genre g munie d'un sous-ensemble fini de points $\Sigma=\{P_1;\dots,P_n\}$ et d'une partition $k=(k_1,k_2,\dots,k_n)\subset(\{-1\}\cup\mathbb{N}^*)^n$ de $4g-4$ possède une structure de surface de demi-translation si il existe un atlas maximal ϕ de cartes allant de $S\backslash\Sigma$ dans des ouverts de $\mathbb{C}\simeq\mathbb{R}^2$ tel que les changements de cartes soient de la forme $z\mapsto\pm z+c$, et tel que pour tout i dans $\{1,\dots,n\}$ il existe un voisinage U_i de P_i, un voisinage V_i de 0 dans $\mathbb{D}_{/\pm}$ et un revêtement ramifié $p=(U_i,P_i)\to(V_i,0)$ de degré k_i+2 tel que chaque restriction injective de p est une carte de ϕ.

Si (S,q) est une surface de demi-translation alors en notant Σ l'ensemble des singularités de q et $(k_1,\dots,k_n)$ leur ordres correspondants, sa structure de surface de demi-translation est donnée par les cartes locales ϕ_P au voisinage $U(P)$ de P point régulier de ω, définies par :

$$\begin{array}{cccc} \phi_P : & U(P) & \to & \mathbb{C} \\ & Q & \mapsto & \int_P^Q q^{1/2} \end{array},$$

où on a choisi une racine carré de q localement au voisinage de P.

Réciproquement une structure de surface de demi-translation sur (S,Σ,d) définit une structure de surface de Riemann sur S et une différentielle quadratique par le tiré en arrière de la forme $\mathrm{d}z^2$ sur $\mathbb{C}$ par les cartes de ϕ.

En plus de cette structure, une surface de demi-translation (S,q) hérite naturellement

- d'une métrique plate à singularités coniques, un zéro d'ordre k correspondant à une singularité d'angle $(k+2)\pi$, et un pôle à une singularité d'angle π (par abus de langage, on pourra voir un pôle comme un zéro d'ordre -1),
- d'une forme d'aire induite par q, la condition sur les pôles garantissant que l'aire de la surface obtenue est finie,
- de champs de directions parallèles privilégié (par convention le champ de directions verticales).

Ces définitions équivalent à la description par recollement de polygones : un polygone correspond tout simplement à une carte bien choisie de ces surfaces, les côtés sont donnés par intégration de ces formes entre deux singularités (zéros ou pôles).

Définition 5. On appellera *lien selle* tout segment géodésique joignant deux singularités (ou la même), et ne traversant aucune autre singularité.

Dans la représentation polygonale les côtés du polygone en particulier représentent des liens selles.

Lien avec les billards polygonaux

Les surfaces plates interviennent dans l'étude de certains billards polygonaux. Les billards polygonaux à angles rationnels peuvent être déployés ce façon à former une surface de translation, telle que le flot linéaire dans cette surface de translation corresponde au flot du billard d'origine. Par exemple sur la Figure 2 le triangle d'angle $\pi/2$, $\pi/8$ et $3\pi/8$ se déploie pour former un octagone régulier représentant une surface de translation possédant une singularité d'angle 6π.

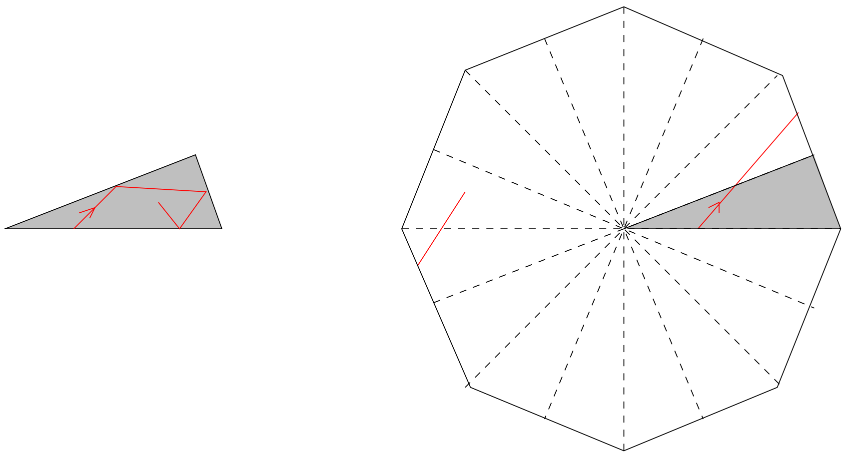


FIGURE 2 – Dépliage de la trajectoire dans un billard triangulaire

Des billards polygonaux à angles multiples de $\pi/2$ peuvent être dupliqués et recollés sur les bords de façon pour former des surfaces de demi-translation dans lesquelles le flot linéaire correspond au flot du billard d'origine. Par exemple sur la Figure 3 on obtient une surface de demi-translation possédant 5 singularités d'angle π et une d'angle 3π.

FIGURE 3 – Dépliage de la trajectoire dans un billard à angle droits

Pour de telles billards l'étude du flot linéaire dans la surface plate correspondante permet de déduire toutes les informations sur la trajectoire dans le billards d'origine. Pour une présentation complète du sujet consulter [Zo4], [MT] et [GJ].

1.2 Espaces de modules de surfaces plates

Nous notons $\mathcal{H}_g$ l'espace de module des différentielles abéliennes, défini comme l'ensemble des surfaces de translation de genre g modulo les classes d'isotopies de difféomorphismes préservant l'orientation, ainsi que $\mathcal{Q}_g$ l'espace de module des différentielles quadratiques.

En déformant légèrement les côtés des polygones précédents, nous obtenons des surfaces plates avec les mêmes singularités coniques, et de même genre. Ces deux données sont liées par l'identité de Gauss-Bonnet. Ainsi, les espaces de modules $\mathcal{H}_g$ et $\mathcal{Q}_g$ sont stratifiés par la donnée des ordres des singularités :

$$\mathcal{H}_g = \bigcup_{\substack{\sum_i d_i = 2g-2 \\ d_i \in \mathbb{N}}} \mathcal{H}(d_1, d_2, \ldots, d_n)$$

où $\mathcal{H}(d_1, d_2, \ldots, d_n)$ désigne l'ensemble des différentielles abéliennes à n zéros de degrés $d_1, \ldots, d_n$ dans l'espace de module $\mathcal{H}_g$. Les strates $\mathcal{H}(d_1, d_2, \ldots, d_n)$ constituent des orbifolds analytiques complexes de dimension $2g + n - 1$.

De la même façon :

$$\mathcal{Q}_g = \bigcup_{\substack{\sum_i k_i = 4g-4 \\ k_i \in \mathbb{N}}} \mathcal{Q}(k_1, k_2, \ldots, k_n)$$

où $\mathcal{Q}(k_1, k_2, \ldots, k_n)$ désigne l'ensemble des différentielles quadratiques à n singularités d'ordre $k_1, \ldots, k_n$ dans l'espace de module $\mathcal{Q}_g$. Les strates $\mathcal{Q}(k_1, k_2, \ldots, k_n)$ constituent des orbifolds analytiques complexes de dimension $2g + n - 2$.

De plus si on note $\mathcal{M}_g$ l'espace de module des surfaces de Riemann de genre g, les espaces $\mathcal{H}_g$ et $\mathcal{Q}_g$ constituent des fibrés au-dessus de $\mathcal{M}_g$, et $\mathcal{Q}_g$ s'identifie au fibré cotangent de $\mathcal{M}_g$.

Les strates de ces espaces de modules ne sont pas toutes connexes : la classification des composantes connexes a été effectuée dans le cas abélien par Kontsevich et Zorich ([KZ]) et dans le cas quadratique par Lanneau ([La1], [La2]).

Dans l'exemple de la Figure 1, la surface de gauche correspond à un point de la strate $\mathcal{H}(2, 0)$ et la surface de droite à un point de la strate $\mathcal{Q}(3, 3, -1, -1)$.

Coordonnées et mesure sur les strates de différentielles abéliennes

Déformer légèrement une surface de translation correspond intuitivement à déformer un peu les côtés du polygone qui la définit. Et en effet, les coordonnées locales dans une strate $\mathcal{H}(d_1, d_2, \ldots, d_n)$ sont données par les périodes relatives, c'est-à-dire les affixes de certains liens selles et géodésiques fermées. Plus précisément, soit (S, ω) une surface de translation dans la strate $\mathcal{H}(d_1, d_2, \ldots, d_n)$. On note $H_1(S, \Sigma, \mathbb{Z})$ le groupe d'homologie relative à l'ensemble $\Sigma = \{P_1, \ldots, P_n\}$ des lieux des zéros de ω. Tout élément de ce groupe peut être représenté par un lien selle ou une géodésique fermée. On note également $H^1(S, \Sigma, \mathbb{C}) \simeq Hom(H_1(S, \Sigma, \mathbb{Z}), \mathbb{C})$ l'espace de cohomologie dual. Il existe un voisinage U de (S, ω) tel que pour tout (S', ω') dans U, on puisse identifier $H^1(S', \Sigma', \mathbb{C})$ à $H^1(S, \Sigma, \mathbb{C})$, en identifiant les sous-réseaux $H^1(S', \Sigma', \mathbb{Z} + i\mathbb{Z})$ et $H^1(S, \Sigma, \mathbb{Z} + i\mathbb{Z})$, c'est-à-dire en utilisant la connexion de Gauss–Manin. Alors les coordonnées locales au voisinage de (S, ω) sont données par l'application période :

$$\begin{array}{rccc} \Theta : & U & \to & H^1(S, \Sigma, \mathbb{C}) \\ & (S', \omega') & \mapsto & \left(\gamma \mapsto \int_\gamma \omega'\right). \end{array}$$

La mesure de Lebesgue en coordonnées définit une mesure μ sur chaque strate $\mathcal{H}(d_1, \ldots, d_n)$.

L'aire d'une surface de translation (S, ω) est donnée par $\frac{i}{2} \int_S \omega \wedge \overline{\omega}$. On notera $\mathcal{H}_g^1$ et $\mathcal{H}_1(d_1, \ldots, d_n)$ les hyperboloïdes des espaces de modules ou des strates formés par les surfaces d'aire 1.

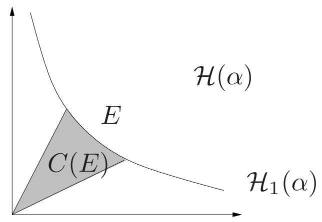
La mesure μ induit une mesure μ_1 sur l'hyperboloïde $\mathcal{H}^1(d_1, \ldots, d_n)$, appelée mesure de Masur–Veech, de la façon suivante : pour tout sous-ensemble E de $\mathcal{H}_1(d_1, \ldots, d_n)$ on définit le cône $C(E)$ au-dessous de E dans $\mathcal{H}(d_1, \ldots, d_n)$ par

$$C(E) = \{(S', \omega') \in \mathcal{H}(d_1, \ldots, d_n), \text{ t.q. } \exists r \in (0, 1], \exists (S, \omega) \in E, (S', \omega') = (S, r\omega)\}.$$

Pour tout sous-ensemble E tels que $C(E)$ soit mesurable, on définit

$$\mu_1(E) = \mu(C(E)).$$

Les strates sont de volume fini par rapport à la mesure μ_1 ([Ma1], [Ve1]).



Coordonnées et mesure sur les strates de différentielles quadratiques

Pour définir de même les coordonnées locales pour les strates de différentielles quadratiques, il faut se ramener au cas abélien en prenant le revêtement double d'orientation de chaque surface de demi-translation.

Soit (S, q) une surface de demi-translation telle que q ne soit pas globalement le carré d'une 1-forme holomorphe. Alors elle admet un revêtement double ramifié canonique $\hat{S} \xrightarrow{p} S$ telle que la différentielle quadratique induite sur $\hat{S}$ soit globalement le carré d'une différentielle abélienne, c'est-à-dire que $p^*q = \omega^2$ avec $(\hat{S}, \omega) \in \mathcal{H}(\hat{\alpha})$. On note Σ l'ensemble des points singuliers de ω et $H^1_-(\hat{S}, \hat{\Sigma}, \mathbb{C})$ l'espace de cohomologie relative anti-invariant par rapport à l'involution naturelle de $(\hat{S}, \omega)$. Dans un voisinage U de (S, q) on peut identifier ces espaces au-dessus de deux surfaces de semi-translation distinctes à l'aide de la connexion de Gauss–Manin. Les coordonnées locales sont alors définies par l'application période suivante :

$$\begin{array}{cccc} \Theta : & U & \to & H^1_-(\hat{S}, \hat{\Sigma}, \mathbb{C}) \\ & (S', q') & \mapsto & \left(\gamma \mapsto \int_\gamma \omega'\right). \end{array}$$

La mesure de Lebesgue en coordonnées locales induit également une mesure de Masur–Veech sur les strates.

De même on peut définir l'aire d'une surface de demi-translation en choisissant une racine carré de q sur une carte maximale, et on note $\mathcal{Q}^1_g$ et $\mathcal{Q}_1(k_1, \ldots, k_n)$ les hyperboloïdes formés par les surfaces d'aire 1/2 (ainsi le revêtement double a pour aire 1).

Comme dans le cas abélien la mesure de Masur–Veech sur $\mathcal{Q}(k_1, \ldots, k_n)$ induit une mesure finie sur $\mathcal{Q}_1(k_1, \ldots, k_n)$.

1.3 Action de $SL(2, \mathbb{R})$ sur les strates

Définition

L'action $SL(2, \mathbb{R})$ sur les surfaces de translation et de demi-translation apparait naturellement quand on les considère comme des polygones tracés dans le plan : en effet il suffit de faire agir $SL(2, \mathbb{R})$ sur chaque côté du polygone.

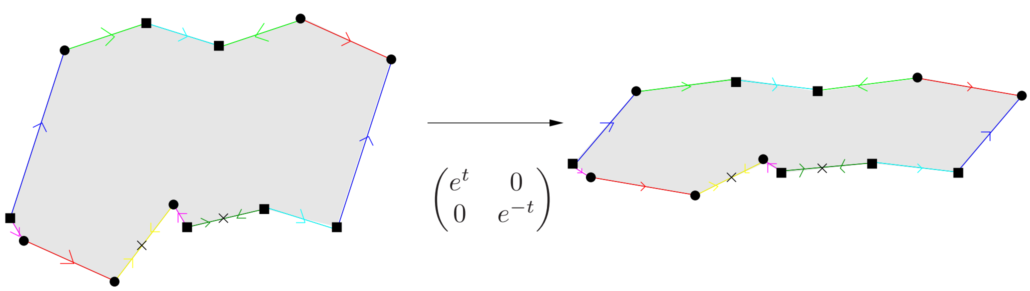


FIGURE 4 – Action d'un élément de $SL(2, \mathbb{R})$

Plus rigoureusement l'action de $SL(2, \mathbb{R})$ sur $\mathcal{H}_g$ est donnée par post-composition dans les cartes de translation : l'action d'un élément $A \in SL(2, \mathbb{R})$ sur $(S, \omega) \in \mathcal{H}_g$ est donnée par $A \cdot (S, \omega) = (S, A \cdot \omega)$ où $A \cdot \omega$ est la différentielle abélienne correspondant à l'atlas de translation $A \circ \phi$ où ϕ est un atlas de translation pour (S, ω).

L'action de $SL(2,\mathbb{R})$ sur $\mathcal{Q}_g$ est définie de façon similaire. Cette action préserve l'aire de la surface, donc se restreint aux hyperboiloïdes $\mathcal{H}_g^1$ et $\mathcal{Q}_g^1$. Elle préserve aussi les ordres des singularités donc se restreint aux strates normalisées $\mathcal{H}^1(d_1,\dots,d_n)$ et $\mathcal{Q}^1(k_1,\dots,k_n)$.

L'action du sous-groupe de $SL(2,\mathbb{R})$ à un paramètre formé par les matrices de la forme $\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$ correspond au flot géodésique par rapport à la métrique de Teichmüller, aussi appelé flot de Teichmüller (voir [Hu] par exemple pour une introduction aux espaces de Teichmüller et la définition de cette métrique).

D'après un résultat fondamental de Masur et Veech le flot de Teichmüller est ergodique par rapport à la mesure de Masur–Veech sur les composantes connexes des strates normalisées ([Ma1], [Ve1]).

L'étude du flot de Teichmüller dans l'espace de module d'une surface plate générique S donne des informations sur le flot linéaire dans la surface S, par un procédé de renormalisation. De plus si la surface provient du déploiement d'un billard polygonal, cela donne des informations sur le flot dans le billard d'origine. Une exemple très explicite de l'utilisation de ce procédé est le modèle d'Ehrenfest du vent dans les arbres étudié par Delecroix, Hubert et Lelièvre ([DHL]), où le taux de diffusion pour le billard de départ est donné par un exposant de Lyapunov du fibré de Hodge le long du flot de Teichmüller dans l'espace de module de la surface plate correspondante.

Courbes de Teichmüller

On s'intéresse aux orbites de l'action de $SL(2,\mathbb{R})$ sur $\mathcal{Q}_g$ (qui s'identifie au fibré cotangent de $\mathcal{M}_g$), et à leur projection sur $\mathcal{M}_g$. Les orbites d'une surface de semi-translation (S,q) sont des copies de $SL(2,\mathbb{R})/SO(2,\mathbb{R})$, qui s'identifie au demi-plan $\mathbb{H}$. Pour presque tout (S,q) la projection de cette orbite sur $\mathcal{M}_g$ est dense.

Notons
$$SL(S,q) = \{g \in SL(2,\mathbb{R}),\ \text{t.q. } g.q = q\}$$
le stabilisateur de (S,q) sous l'action de $SL(2,\mathbb{R})$, appelé groupe de Veech. Alors dans le cas particulier où $SL(S,q)$ est un réseau dans $SL(2,\mathbb{R})$, l'orbite de (S,q) se projette dans $\mathcal{M}_g$ en une courbe algébrique appelée *courbe de Teichmüller*.

De manière équivalente une courbe de Teichmüller est une courbe algébrique complexe de genre $g \geq 2$, donc munie de la métrique hyperbolique donnée par uniformisation, plongée isométriquement dans $\mathcal{M}_g$ muni de la métrique de Teichmüller. Autrement dit, les courbes de Teichmüller correspondent aux courbes complexes totalement géodésiques de $\mathcal{M}_g$.

Les courbes de Teichmüller sont les premiers exemples de lieux $SL(2,\mathbb{R})$ invariants dans les strates d'espaces de modules de surfaces plates, excepté les strates elles-mêmes. L'étude des lieux $SL(2,\mathbb{R})$-invariants a fait l'objet de beaucoup de recherches et de récents développements : en particulier le résultat fondamental de Eskin, Mirzakhani et Mohammadi ([EMi], [EMM]) stipule que l'adhérence de la $GL^+(2,\mathbb{R})$-orbite d'une surface de translation est une sous-variété affine invariante. Les coefficients des équations linéaires définissant cette variétés peuvent être prises dans un corps de nombre ([Wr2]). De plus les sous-variété affines invariantes des strates de différentielles abéliennes sont des sous-variétés algébriques, définies sur $\overline{\mathbb{Q}}$ ([Fi]), ce qui était déjà connu pour les courbes de Teichmüller par les travaux de McMullen et Möller.

1.4 Collections rigides de liens selles

Différentielles abéliennes

On s'intéresse maintenant aux familles de liens selles d'une surface de translation $(S,\omega) \in \mathcal{H}(\alpha)$, et leur image par une petite déformation. Considérons l'exemple donné par la figure 5. On a représenté ici une petite déformation de la surface de gauche, donnée intuitivement par une petite déformation des côtés du polygone. On remarque alors que les liens selles γ_1 (rouge) et γ_2 (vert) restent nécessairement parallèles de même longueur. On dira que la collection formée par ces deux liens selles est *rigide*. Eskin, Masur et Zorich ont remarqué et étudié ce phénomène dans [EMZ].

Pour formaliser cette notion nous donnons la définition suivante :

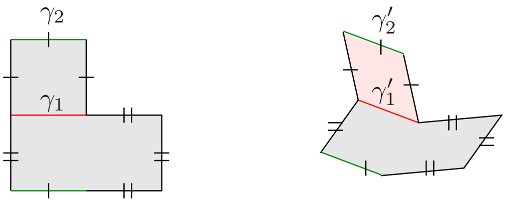


FIGURE 5 – Petite déformation d'une surface de translation

Définition 6. Soit $\mathcal{V}$ une sous-variété $SL(2,\mathbb{R})$-invariante de $\mathcal{H}(\alpha)$, et soit $(S,\omega) \in \mathcal{V}$. Une collection $\{\gamma_1,\dots,\gamma_r\}$ de liens selles sur (S,ω) est dite *rigide dans* $\mathcal{V}$ si toute déformation assez petite de (S,ω) dans $\mathcal{V}$ préserve les proportions $|\gamma_1| : |\gamma_2| : \dots : |\gamma_r|$.

Si $\mathcal{V} = \mathcal{H}(\alpha)$, les coordonnées locales de $\mathcal{H}(\alpha)$ étant données par les périodes relatives, on a naturellement le résultat suivant :

Proposition 1 (Eskin, Masur, Zorich)**.** *Une collection $\{\gamma_1,\dots,\gamma_r\}$ de liens selles sur $(S,\omega) \in \mathcal{H}(\alpha)$ est rigide dans $\mathcal{H}(\alpha)$ si et seulement si ils sont deux à deux homologues.*

Remarquons que dans l'exemple de la figure 5, les liens selles γ_1 et γ_2 sont homologues (ils coupent la surface en deux composantes connexes), et bordent un cylindre formé de géodésiques fermées parallèles qui leur sont homologues. La partie restante de la surface est formée d'un tore percé de deux trous formant une figure en huit, auquel est recollé le cylindre décrit précédemment (cf Figure 6).

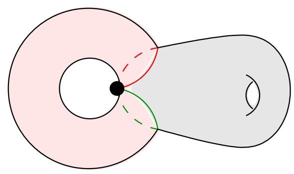


FIGURE 6 – Représentation topologique

La donnée de ces éléments détermine ce qu'on appellera une *configuration* de liens selles homologues :

Définition 7. Une *configuration* de liens selles homologues sur (S,ω) est un des types géométriques possibles des collections maximales de liens selles homologues sur (S,ω).

Les configurations de liens selles homologues ont été étudiées et classifiées par Eskin, Masur et Zorich dans [EMZ].

Différentielles quadratiques

On peut de même considérer les variations de liens selles sur une surface de demi-translation $(S,q) \in \mathcal{Q}(\beta)$ sous petite déformation dans $\mathcal{Q}(\beta)$. A titre d'exemple, sur la Figure 7, les liens selles γ_1 (vert) , γ_2 (rouge) et γ_3 (bleu) restent parallèles et de même longueur sous petite déformation dans la strate.

On a toujours la définition de collection rigide de liens selles :

Définition 8. Soit $\mathcal{V}$ une sous-variété $SL(2,\mathbb{R})$-invariante de $\mathcal{Q}(\beta)$, et soit $(S,q) \in \mathcal{V}$. Une collection $\{\gamma_1,\dots,\gamma_r\}$ de liens selles sur (S,q) est dite *rigide dans* $\mathcal{V}$ si toute déformation assez petite de (S,q) dans $\mathcal{V}$ préserve les proportions $|\gamma_1| : |\gamma_2| : \dots : |\gamma_r|$.

Si $\mathcal{V} = \mathcal{Q}(\beta)$, les coordonnées locales étant données par la partie anti-invariante de l'homologie dans le revêtement double, on introduit les notations suivantes : si γ est un lien selle (ou une géodésique fermée simple) de (S,q), on note γ' et γ'' ses pré-images dans le revêtement double $(\hat{S},\omega)$. Puis :

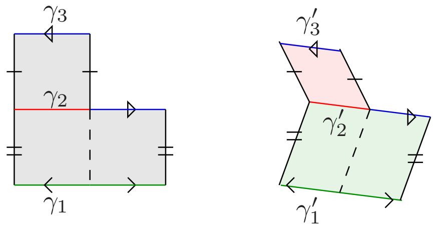


FIGURE 7 – Petite déformation d'une surface de demi-translation

- si $[\gamma] \not\equiv 0$, on note $[\hat{\gamma}] = [\gamma'] - [\gamma''] \in H_1^-(\hat{S}, \hat{\Sigma}, \mathbb{Z})$,
- sinon, on note $[\hat{\gamma}] = [\gamma']$.

On obtient ainsi des éléments de $H_1^-(\hat{S}, \hat{\Sigma}, \mathbb{Z})$, dont le dual complexe donne des coordonnées locales sur $\mathcal{Q}(\beta)$ au voisinage de (S, q).

Définition 9 (Masur, Zorich)**.** Deux liens selles γ_1 et γ_2 sur (S, q) sont dits *ĥomologues* si $[\hat{\gamma}_1] = [\hat{\gamma}_2]$ dans $H_1^-(\hat{S}, \hat{\Sigma}, \mathbb{Z})$.

Les collections rigides de liens selles dans $\mathcal{Q}(\beta)$ sont caractérisées par le résultat suivant :

Proposition 2 (Masur, Zorich)**.** *Une collection $\{\gamma_1, \ldots, \gamma_r\}$ de liens selles sur $(S, q) \in \mathcal{Q}(\beta)$ est rigide dans $\mathcal{Q}(\beta)$ si et seulement si ils sont deux à deux ĥomologues.*

Dans l'exemple de la Figure 7, la famille $\{\gamma_1, \gamma_2, \gamma_3\}$ est rigide dans $\mathcal{Q}(2, -1^2)$.

Remarquons de plus que les liens selles γ_2 (rouge) et γ_3 (bleu) bordent un cylindre formé de géodésiques fermées parallèles, qui leur sont ĥomologues. De même le lien selle γ_1 (vert) borde un cylindre formé de géodésiques fermées parallèles qui lui sont ĥomologues (bien qu'étant homologues à 0), de longueur 2 fois plus grande que pour le cylindre précédent, et qui est bordé de l'autre côté par la réunion des liens selles γ_2 et γ_3 (cf Figure8).

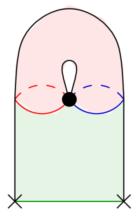


FIGURE 8 – Représentation topologique

La notion d'être ĥomologue est subtile : en effet dans l'exemple γ_1 et γ_2 sont de natures différentes puisque γ_1 relie deux pôles distincts et γ_2 relie un zéro d'ordre 2 à lui-même, et ne sont pas homologues, elles sont pourtant ĥomologues. De même on peut avoir une courbe homologue à zéro mais pas ĥomologue à 0.

Comme précédemment, on a :

Définition 10. Une *configuration* de liens selles ĥomologues sur (S, q) est un des types géométriques possibles des collections maximales de liens selles ĥomologues sur (S, q).

Les configurations de liens selles ĥomologues ont été étudiées et classifiées pour les strates par Masur et Zorich dans [MZ], et pour les composantes hyperelliptiques de strates par Boissy dans [Bo].

Nous nous intéresserons particulièrement aux configurations de liens selles homologues (ou ĥomologues selon le contexte), telles qu'il y ait dans la collection au moins certains liens selles bordant un cylindre formé de géodésiques fermées parallèles. Nous parlerons alors de *configuration à cylindres*. Les études de [EMZ] et [MZ] montrent que les configurations à cylindres correspondent aux configurations de géodésiques fermées simples homologues (ou ĥomologues suivant le contexte).

1.5 Constantes de Siegel–Veech

Les constantes de Siegel–Veech donnent l'asymptotique du nombre de géodésiques fermées ou de liens selles dans une surface plate. Pour les surfaces plates provenant de billards polygonaux par dépliage ou recollage, le nombre de géodésique fermées donne le nombre de trajectoires périodiques dans le billard d'origine.

Soit S une surface plate représentant soit une surface de translation (S,ω) soit une surface de demi-translation (S,q). Nous introduisons les quantités suivantes :

- $N_{l.s.}(S,L)$ le nombre de liens selles de longueur inférieure à L sur S
- $N_{g.f.}(S,L)$ le nombre de familles géodésiques fermées simples de longueur inférieure à L sur S : notons qu'une géodésique fermée simple régulière (ne passant pas par des singularités) dans une surface plate fait toujours partie d'un cylindre plat formé de géodésiques fermées qui lui sont parallèles et qui sont de même longueur. $N_{g.f.}(S,L)$ compte le nombre de cylindres formés par des géodésiques fermées de longueur inférieure à L.

Alors si S est générique (c'est-à-dire pour presque toute surface plate S dans la strate relativement à la mesure de Masur–Veech), d'après les résultats de Eskin et Masur ([EMa]) la limite

$$\lim_{L\to\infty}\frac{N_*(S,L)\cdot aire(S)}{\pi L^2}=c_*$$

est bien définie et ne dépend pas de la surface S pour presque toute S dans la strate normalisée.

La constante c_* est appelée constante de Siegel–Veech pour la strate de S. Par exemple pour le tore plat $\mathbb{T}=\mathbb{C}/(\mathbb{Z}+i\mathbb{Z})$, $N_{g.f.}(\mathbb{T},L)$ représente le nombre de points primitifs du réseau $\mathbb{Z}+i\mathbb{Z}$ dans la boule de centre 0 et de rayon L dans $\mathbb{C}$, on obtient alors par des considérations arithmétiques :

$$c_{g.f.}(\mathbb{T})=c_{g.f.}(\mathcal{H}(0))=\frac{1}{2\zeta(2)}=\frac{3}{\pi^2}.$$

Historiquement Masur en 1988-1990 a montré ([Ma2], [Ma3]) un encadrement quadratique en L de $N_*(S,L)$ pour chaque surface S de la strate, puis Veech en 1998 ([Ve3]) a montré qu'en moyennant $N_*(S,L)$ sur la composante connexe de la strate normalisée correspondante on obtenait une expression quadratique en L, pour tout L, en enfin Eskin et Masur en 2001 ([EMa]) ont montré l'asymptotique quadratique en L de $N(S,L)$ pour presque toute S dans la strate, et on montré que la constante correspondante de dépendait pas de S.

Tous ces résultats sont montrés dans un cadre un peu plus général : en effet on peut compter les liens selles ou les géodésiques avec poids, à condition que le poids vérifie certaines conditions, en particulier qu'il soit $SL(2,\mathbb{R})$ invariant. Par exemple on peut compter les géodésiques fermées avec pour poids l'aire des cylindres correspondants : on note $N_{area}(S,L)$ et c_{area} la fonction de comptage et la constante de Siegel–Veech correspondante.

On peut également compter les liens selles (ou les géodésiques fermées) uniquement quand elles forment une configuration $\mathcal{C}$ (au sens du paragraphe précédent), et alors les constantes de Siegel–Veech pour les strates sont la somme sur les configurations $\mathcal{C}$ admissibles pour la strate des constantes de Siegel–Veech pour $\mathcal{C}$.

Eskin, Masur et Zorich ont développé dans [EMZ] des techniques pour calculer les constantes de Siegel–Veech pour les configurations dans le cas abélien (rappelées dans les chapitres 2 et 3), qui ont été généralisées dans le cas quadratique en genre 0 par Athreya, Eskin et Zorich ([AEZ1]). Dans cette thèse nous généralisons le résultat dans el cas quadratique en genre supérieur (chapitre 3).

Les constantes de Siegel–Veech c_{area} pour les strates relativement à l'aire des cylindres sont particulièrement importantes car elles sont reliées très explicitement à la somme des exposants de Lyapunov du fibré de Hodge le long du flot de Teichmüller par la formule clé de Eskin–Kontsevich–Zorich ([EKZ2]). Les exposants de Lyapunov donnent le spectre de déviation de feuilletages mesurés sur des surfaces plates (voir [Fo1], [Fo2], [Zo1], [Zo2]), ce qui permet des applications aux billards polygonaux, aux transformations d'échanges d'intervalles, etc.

Ces constantes sont de plus en plus étudiées, de différents points de vue ([AEZ1], [Ba1], [Ba2], [BG], [EKZ2], [Vo]).

1.6 Volumes de strates d'espaces de modules de surfaces plates

Les techniques de [EMZ] relient les constantes de Siegel–Veech pour les configurations aux volumes de certaines strates, ou de certaines composantes connexes de strates.

Les volumes des strates de différentielles abéliennes ont été calculés par Eskin et Okounkov ([EOk1]) en utilisant des formes modulaires et la théorie des représentations. Dans le cas quadratique, une étude similaire ([EOPa]) ne permet pas d'obtenir facilement des valeurs explicites pour les volumes. L'article [AEZ2] calcule explicitement les volumes des strates de différentielles quadratiques à zéros et pôles simples en genre 0, en s'appuyant sur une formule de Kontsevich ([Ko1]).

L'idée commune à tous ces travaux est d'estimer le nombre de "points entiers" dans la strate pour en déduire les volumes. Ces points entiers correspondent à des surfaces plates représentées par les polygones à petits carreaux. Le chapitre 4 rappelle ce lien est expose quelques estimations de volumes de strates de petite dimension, où un calcul "à la main" est possible.

2 Récapitulatif des résultats présentés

Chaque chapitre de la thèse développe une des thématiques décrites dans les sections suivantes. Ils prennent la forme d'articles. Le premier correspond à un article [Go1] publié dans *Mathematical Research Letters*. Le deuxième correspond à une prépublication [BG], et enfin les deux derniers chapitres correspondent à deux parties d'une prépublication [Go2]. Nous résumons ici les principaux résultats exposés dans la suite, en donnant pour chaque résultat la notation correspondante intervenant dans les chapitres en anglais.

2.1 Courbes de Teichmüller

Dans cette partie nous nous intéressons à la question suivante : comment construire de nouvelles courbes de Teichmüller, et à cette fin, comment détecter qu'une courbe est une courbe de Teichmüller ?

Rappelons que si (S, ω) est une surface de translation, alors la projection de son orbite sous l'action de $SL(2, \mathbb{R})$ sur l'espace de modules de surfaces de Riemann $\mathcal{M}_g$ est appelée *courbe de Teichmüller*, à condition que le stabilisateur de la surface pour cette action soit un réseau dans $SL(2, \mathbb{R})$. Les courbes de Teichmüller sont les premiers exemples de lieux $SL(2, \mathbb{R})$-invariants dans $\mathcal{H}_g$. Ceci est valable aussi pour les surfaces de demi-translation dans l'espace $\mathcal{Q}_g$ (cf section1.3).

Plusieurs critères algébriques sont connus pour les courbes de Teichmüller, comme celui décrit par Bouw et Möller dans [BM]. Le chapitre 1 ([Go1]) décrit un critère, déjà formulé sous une forme un peu différente par Martin Möller, en terme de fibré maximal Higgs. On en donne une preuve utilisant des résultats de dynamique.

Le critère est le suivant.

Pour une surface de Riemann S, il existe une forme pseudo-hermitienne naturelle sur $H^1(S, \mathbb{C})$, qui est définie positive sur $H^{1,0}(S, \mathbb{C})$. Cette forme induit une forme sur le fibré de Hodge H^1 au-dessus de l'espace de modules $\mathcal{M}_g$ des surfaces de Riemann de genre g, où la fibre au-dessus d'un point S est $H^1(S, \mathbb{C})$. La connexion de Gauss–Manin préserve cette forme pseudo-hermitienne.

Soit $\mathcal{C}$ une courbe complexe dans $\mathcal{M}_g$, de genre $g \geq 1$.

Le théorème de semi-simplicité de Deligne implique que le fibré de Hodge au-dessus de $\mathcal{C}$ se décompose en somme directe orthogonale de sous-fibrés plats, tels que la restriction de la forme pseudo-hermitienne canonique sur chaque sous-fibré est non dégénérée. Supposons qu'un des sous-fibré $\mathbb{L}$ dans la décomposition soit de rang r et tel que la signature de la restriction de la forme pseudo-hermitienne soit $(1, r-1)$. Définissons $\mathbb{L}^{1,0} = \mathbb{L} \cap H^{1,0}$ et $\mathbb{L}^{0,1} = \mathbb{L} \cap H^{0,1}$. Alors $\mathbb{L}^{1,0}$ est un fibré en droites holomorphe au-dessus de $\mathcal{C}$.

Soit $\overline{\mathbb{L}^{1,0}}$ l'extension de Deligne du fibré $\mathbb{L}^{1,0}$ au bord de l'espace de modules, et $\chi(\mathcal{C})$ la caractéristique d'Euler généralisée de $\mathcal{C}$.

Théorème 1 (Theorem 1). *Si $\chi(\mathcal{C}) \geq 0$, alors $\mathcal{C}$ n'est pas une courbe de Teichmüller.*

Supposons $\chi(\mathcal{C}) < 0$. Pour tout sous-fibré plat $\mathbb{L}$ du fibré de Hodge sur $\mathcal{C}$ satisfaisant les hypothèses précédentes, on a :

$$\deg \overline{\mathbb{L}^{1,0}} \leq -\frac{\chi(\mathcal{C})}{2}. \tag{1}$$

Si il y a égalité, alors $\mathcal{C}$ est une courbe de Teichmüller, et le fibré en droites $\mathbb{L}^{1,0}$ est le fibré tautologique.

Toute courbe de Teichmüller correspondant à une strate de différentielles abéliennes admet un sous-fibré plat $\mathbb{L}$ du fibré de Hodge vérifiant les conditions précédentes, tel que :

$$\deg \overline{\mathbb{L}^{1,0}} = -\frac{\chi(\mathcal{C})}{2}.$$

Ce résultat peut être reformulé comme suit : si le sous-fibré plat $\mathbb{L}$ a un exposant de Lyapunov égal à 1, alors la courbe est de Teichmüller. La preuve de ce théorème utilise une comparaison de métriques possible grâce à la notion de métrique de Kobayashi, et une estimation de la variation de la norme de Hodge le long du flot géodésique due à Giovanni Forni.

Dans [BM], Bouw et Möller donnent un critère similaire qu'ils appliquent à une famille de courbes choisie de façon à ce que le groupe affine de la courbe de Teichmüller obtenue soit le groupe triangulaire (n, ∞, ∞).

2.2 Géométrie des configurations à cylindres

Nous nous intéressons dans cette section aux configurations à cylindres de liens selles homologues sur une surface de translation, ou de façon équivalente aux configurations de géodésiques fermées homologues. En effet ces configurations sont cruciales dans le calcul des constantes de Siegel–Veech pour la strate correspondante relativement à l'aire des cylindres.

Avec Max Bauer nous avons exploré la géométrie de ces configurations, afin d'obtenir des informations quantitatives relatives aux cylindres des configurations.

Nous définissons les fonctions de comptage suivantes. Si S est une surface de translation dans la strate $\mathcal{H}(\alpha)$, $\mathcal{C}$ une configuration de liens selles homologues admissible sur S, et L un réel positif, nous notons

- $N_{conf}(S, \mathcal{C}, L)$ le nombre de configurations de liens selles de type $\mathcal{C}$ sur S, de longueur inférieure à L,
- $N_{cyl}(S, \mathcal{C}, L)$ le nombre de configurations comme précédemment comptées avec poids le nombre de cylindres,
- $N_{area^p}(S, \mathcal{C}, L)$ le nombre de configurations comme précédemment comptées avec poids la somme des aires de chaque cylindre à la puissance p,
- $N_{area^p,\, conf}(S, \mathcal{C}, L)$ le nombre de configurations comme précédemment comptées avec poids la puissance p-ième de l'aire totale occupée par les cylindres,
- $N_{conf,\, A_1 \geq p}(S, \mathcal{C}, L)$ le nombre de configurations de type $\mathcal{C}$ sur S, de longueur inférieure à L, telles que le premier cylindre dans la configuration (il y a une façon déterminée de les numéroter) occupe au moins un proportion p de l'aire de la surface totale,
- $N_{conf,\, A \geq p}(S, \mathcal{C}, L)$ le nombre de configurations de type $\mathcal{C}$ sur S, de longueur inférieure à L, telles que l'aire totale des cylindres occupe au moins une partie p de l'aire totale de la surface.

À chacune de ces fonctions de comptage nous associons la constante de Siegel–Veech correspondante, définie par la formule

$$c_*(\mathcal{C}) = \lim_{L \to \infty} \frac{N_*(S, \mathcal{C}, L) \cdot aire(S)}{\pi L^2}$$

pour presque surface S dans la strate $\mathcal{H}(\alpha)$. Ces constantes sont bien définies par le résultat de Eskin et Masur ([EMa]).

Nous considérons les rapports suivants :

- $\dfrac{c_{area^p}(\mathcal{C})}{c_{cyl}(\mathcal{C})}$, qui peut être interprété comme l'aire "moyenne" d'un cylindre à la puissance p dans la configuration $\mathcal{C}$,

- $\dfrac{c_{area^p,\ conf}(\mathcal{C})}{c_{conf}(\mathcal{C})}$, qui peut être interprété comme l'aire moyenne de la partie périodique de la surface à la puissance p pour la configuration $\mathcal{C}$,

- $\dfrac{c_{area,\ A_1\geq p}(\mathcal{C})}{c_{conf}(\mathcal{C})}$, qui peut être interprété comme la proportion de surfaces ayant un large cylindre pour la configuration $\mathcal{C}$,

- $\dfrac{c_{area,\ A_1\geq p}(\mathcal{C})}{c_{conf}(\mathcal{C})}$, qui peut être interprété comme la proportion de surfaces ayant une large partie périodique pour la configuration $\mathcal{C}$.

Alors nous montrons le résultat suivant :

Théorème 2 (Theorems 2, 3, 4, 5)**.** *Soit $\mathcal{H}(\alpha)$ une strate de différentielles abéliennes et $\mathcal{C}$ une configuration de liens selles admissibles pour $\mathcal{H}(\alpha)$ contenant q cylindres. Alors nous avons :*

$$\begin{aligned}
\frac{c_{area^p}(\mathcal{C})}{c_{cyl}(\mathcal{C})} &= \frac{(d-2)!}{(p+1)\cdot(p+2)\cdots(p+d-2)}\\
\frac{c_{area^p,\ conf}(\mathcal{C})}{c_{conf}(\mathcal{C})} &= \frac{q\cdot(q+1)\cdots(q+n-1)}{(p+Q)\cdot(p+Q+1)\cdots(p+q+n-1)}\\
\frac{c_{area,\ A_1\geq p}(\mathcal{C})}{c_{conf}(\mathcal{C})} &= (1-p)^{d-2}\\
\frac{c_{area,\ A\geq p}(\mathcal{C})}{c_{conf}(\mathcal{C})} &= \frac{B(1-p;n,q)}{B(n,q)}
\end{aligned}$$

où d est la dimension complexe de $\mathcal{H}(\alpha)$, $n = d-q-1$, et $B(x;a,b)$ représente la fonction Beta incomplète donnée par

$$B(x;a,b) = \int_0^x u^{a-1}(1-u)^{b-1}du$$

et $B(a,b) = B(x;a,b)$.

Enfin nous nous intéressons aux configurations extrémales, et en particulier celles qui maximisent l'aire moyenne de la partie périodique, donnée par le rapport

$$\frac{c_{area}(\mathcal{C})}{c_{conf}(\mathcal{C})} =: c_{mean\ area\ conf}(\mathcal{C}).$$

Pour cela nous avons étudié les configurations contenant le maximum de cylindres.

Pour les strates principales le rapport précédent atteint la valeur maximale $\frac{1}{4}$, alors que pour les composantes hyperelliptiques, $\mathcal{H}^{hyp}(g-1,g-1)$ par exemple, il atteint la valeur $\frac{1}{2g}$. Finalement nous montrons :

Théorème 3 (Theorem 7)**.** *Pour K une composante connexe de $\mathcal{H}(\alpha)$, et pour toute configuration $\mathcal{C}$ admissible pour K, l'aire moyenne asymptotique de la partie périodique pour la configuration $\mathcal{C}$ vérifie*

$$c_{mean\ area\ conf}(\mathcal{C}) \leq \frac{1}{3}$$

et cette bonne supérieure est atteinte pour $K = \mathcal{H}^{odd}(2,2,\ldots,2)$.

2.3 Constantes de Siegel–Veech pour les différentielles quadratiques

Nous nous intéressons maintenant à une nouvelle application des constantes de Siegel–Veech. En effet rappelons qu'elles sont reliées à la dynamique du flot de Teichmüller sur l'espace de modules des différentielles abéliennes ou quadratiques, par une formule prouvée par Eskin Kontsevich et Zorich dans [EKZ2], qui exprime la somme des exposants de Lyapunov pour le cocycle de Kontsevich–Zorich en fonction de la constante c_{area} de la strate considérée.

Dans [EMZ], Eskin Masur et Zorich expriment la constante de Siegel–Veech pour une configuration donnée en terme de volumes de strates dans lesquelles vivent les surfaces plus petites obtenues en découpant la surface de départ le long des géodésiques homologues. Ainsi ils ont montré une formule explicite pour chaque configuration dans le cas abélien. Athreya Eskin et Zorich ont fait le même travail dans le cas quadratique pour le genre 0 ([AEZ1]). Ils ont même obtenu ainsi les valeurs des volumes des strates en genre 0, car les constantes de Siegel–Veech c_{area} sont connues en genre 0, grâce au théorème d'Eskin–Kontsevich–Zorich.

Les résultats présentés dans cette partie sont une généralisation des résultats en genre 0 de [AEZ1] et des résultats pour les différentielles abéliennes de [EMZ]. De façon analogue à la partie précédente on peut définir des constantes de Siegel–Veech $c_{conf}(\mathcal{C})$, $c_{cyl}(\mathcal{C})$ et $c_{area}(\mathcal{C})$ pour les configurations de liens selles ĥomologues. Ces configurations sont beaucoup plus complexes que dans le cas abélien, en particulier, les cylindres bordés par des liens selles ĥomologues peuvent être de même largeur ou de largeur en proportion double (voir Figure 7 pour un exemple). Nous parlons alors de cylindres "fins" ou "épais", les deuxièmes étant deux fois plus larges que les premiers. L'autre différence avec le cas quadratique est que la strate de bord, c'est-à-dire la strate à laquelle appartient la partie complémentaire de la surface obtenue après suppression des liens selles ĥomologues et de leurs cylindres associés, peut être vide : dans ce cas la surfaces est uniquement constituée de cylindres bordés par des liens selles ĥomologues (c'est le cas dans l'exemple de la Figure 7).

Nous obtenons le résultat suivant :

Théorème 4 (Theorem 8). *Soit $\mathcal{C}$ une configuration de liens selles ĥomologues admissible pour une strate $\mathcal{Q}(\alpha)$ de différentielles quadratiques. Notons q_1 le nombre de cylindres fins et q_2 le nombre de cylindres épais. Si $q = q_1 + q_2$ est au moins égal à 1 et que la strate de bord $\mathcal{Q}(\alpha')$ est non vide, alors :*

$$c(\mathcal{C}) = \frac{M}{2^{q+2}} \frac{(\dim_{\mathbb{C}} \mathcal{Q}(\alpha') - 1)!}{(\dim_{\mathbb{C}} \mathcal{Q}(\alpha) - 2)!} \frac{\operatorname{Vol} \mathcal{Q}_1(\alpha')}{\operatorname{Vol} \mathcal{Q}_1(\alpha)} \tag{2}$$

$$c_{cyl}(\mathcal{C}) = \left(q_1 + \frac{1}{4} q_2\right) c(\mathcal{C}) \tag{3}$$

$$c_{area}(\mathcal{C}) = \frac{1}{\dim_{\mathbb{C}} \mathcal{Q}(\alpha) - 1} c_{cyl}(\mathcal{C}) \tag{4}$$

où M est une constante combinatoire explicite ne dépendant que de la configuration.

Notons qu'il y a des difficultés additionnelles dans le cas quadratique pour calculer de telles constantes : le choix de normalisation pour le volume, les symétries dans les configurations, le fait d'avoir des cylindres de taille différentes, tout cela joue à chaque étape des calculs dans la formule donnant la constante combinatoire M. Il était donc nécessaire de vérifier ces calculs, et pour cela il fallait disposer de valeurs de volumes de strates. C'est l'objet de la partie suivante.

2.4 Volumes de strates de différentielles quadratiques

La principale difficulté pour vérifier les formules données dans le chapitre précédent est que les valeurs des volumes des strates d'espaces de modules de différentielles quadratiques ne sont pas connues : Eskin, Okounkov et Pandharipande ([EOPa]) ont trouvé des séries génératrices dont il est très difficile d'extraire des valeurs même approchées pour des strates même en genre petit. Plusieurs personnes travaillent en ce moment à obtenir de telle valeurs.

C'est pourquoi il était intéressant de calculer quelques unes de ces valeurs en utilisant la méthode décrite pour le genre 0 dans [AEZ2]. Plus précisément nous avons généralisé cette méthode en genre plus grand. Pour l'expliquer donnons une idée du principe de base utilisé pour estimer ces volumes.

De la même façon qu'on peut estimer le volume inclus dans un contour dessiné sur une feuille quadrillée, en comptant le nombre des carreaux inscrits dans cette figure, pour des carreaux assez petits, on peut évaluer les volumes de strates d'espaces de modules en comptant les points "entiers" dans ces espaces. Ici les points entiers correspondent à ce qu'on appelle les surfaces à petits carreaux : ce sont des surfaces plates obtenues par recollement de carrés isométriques le long de côtés parallèles. Ce sont naturellement des revêtements du tore. Eskin et Okounkov ont compté ces revêtements par des méthodes de théorie des représentations. Mais on peut utiliser une approche plus pragmatique, qui consiste à remarquer qu'une surface à petits carreaux se décompose en cylindres horizontaux, recollés entre eux d'une certaine façon. En comptant le nombre de cylindres et le nombre de façon de les recoller entre eux partant d'un nombre fixé de carreaux, on peut alors compter ces surfaces à petits carreaux. C'est la méthode décrite dans [AEZ2]. Le fait de compter les façons de recoller des cylindres fait appel à la théorie des graphes en ruban, aussi appelés *cartes* en combinatoire. Elles ont été beaucoup étudiées mais pour l'instant il n'existe pas de série génératrice pour leur nombre. Heureusement en petit genre et petite complexité elles sont entièrement décrites dans [JV], ce qui permet de calculer certains volumes de strates de petites dimensions. Souhaitons que ces valeurs de volumes servent de valeurs tests pour les futurs possibles algorithmes qui donneront des valeurs numériques approchées des volumes.

D'autre part pour vérifier les formules de la partie précédente il est intéressant de travailler sur les composantes hyperelliptiques de strates : en effet pour celles-là on peut calculer simplement le volumes, et comme il n'y a pas beaucoup de configurations (cf [Bo]) on peut appliquer les formules précédentes et vérifier qu'elles sont cohérentes avec les valeurs des constantes de Siegel–Veech données pour les composantes entières dans [EKZ2].

Chapter 1

A criterion for being a Teichmüller curve

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1.1 Introduction

Given a curve in the moduli space of Riemann surfaces, we want to know whether it is a Teichmüller curve. By Deligne semisimplicity theorem the Hodge bundle over the curve decomposes into a direct sum of flat subbundles admitting variations of complex polarized Hodge structures of weight 1. Suppose that the restriction of the canonical pseudo-Hermitian form to one of the blocks of the decomposition has rank $(1, r-1)$. We establish an upper bound for the degree of the corresponding holomorphic line bundle in terms of the (orbifold) Euler characteristic of the curve. Our criterion claims that if the upper bound is attained, the curve is a Teichmüller curve.

For those Teichmüller curves which correspond to strata of *Abelian* differentials our criterion is necessary and sufficient in the sense that if the curve is a Teichmüller curve, then the decomposition of the Hodge bundle necessarily contains a nontrivial block of rank $(1, 1)$ corresponding to the tautological line bundle for which the upper bound is attained.

The original criterion in the same spirit was found by Martin Möller in [Mö06, Th. 2.13 and 5.3] where the condition detecting a Teichmüller curve is formulated in terms of Higgs bundle, or equivalently in terms of non-vanishing of the second fundamental form (Kodaira-Spencer map). In [Wr1], A. Wright gives an alternative version of Möller's criterion, in terms of non-vanishing of the period map.

The key idea of our criterion is based on Forni's observation that the tautological bundle on a Teichmüller curve is spanned by those vectors of the Hodge bundle which have the maximal variation of the Hodge norm along the Teichmüller flow.

We combine this result of Forni with the Bouw–Möller version of the Kontsevich formula for the sum of the Lyapunov exponents of the Hodge bundle along the Teichmüller geodesic flow. Similar to the criteria mentioned above, the fact that the Teichmüller metric coincides with the Kobayashi metric will be crucial for the proof.

1.2 Criterion

Having a Riemann surface X, the natural pseudo-Hermitian intersection form on $H^1(X,\mathbb{C})$, is defined on closed 1-forms representing cohomology classes as:

$$(\omega_1,\omega_2)=\frac{i}{2}\int_X \omega_1\wedge\overline{\omega_2}\,.$$

Restricted to $H^{1,0}(X,\mathbb{C})$, the form is positive-definite, and restricted to $H^{0,1}(X,\mathbb{C})$, the form is negative-definite.

This pseudo-Hermitian form of signature (g,g) induces a form on the Hodge bundle H^1 over the the moduli space $\mathcal{M}_g$ of Riemann surfaces of genus g, where the fiber H^1_X of the Hodge bundle over a point X in $\mathcal{M}_g$ is $H^1(X,\mathbb{C})$. The pseudo-Hermitian form is covariantly constant with respect to the Gauss–Manin flat connexion on the Hodge bundle.

Let $\mathcal{C}$ be a complex curve in $\mathcal{M}_g$. We want to detect, whether $\mathcal{C}$ is a Teichmüller curve or not. Throughout this paper we assume that the genus g is strictly greater than 1 since in genus one the problem becomes trivial: the moduli space $\mathcal{M}_{1,1}$ is a complex curve itself.

By Deligne semisimplicity theorem [De, Prop. 1.13.], the Hodge bundle over $\mathcal{C}$ splits into a direct sum of orthogonal flat subbundles, such that the restriction of the canonical pseudo-Hermitian form to each subbundle is nondegenerate. Assume that this splitting contains a flat subbundle $\mathbb{L}$ of rank r, where $r\geq 2$, such that the signature of the canonical pseudo-Hermitian form restricted to $\mathbb{L}$ is $(1,r-1)$. Let us define $\mathbb{L}^{1,0}=\mathbb{L}\cap H^{1,0}$ and $\mathbb{L}^{0,1}=\mathbb{L}\cap H^{0,1}$. Deligne semisimplicity theorem combined with our assumption on the signature implies that $\mathbb{L}^{1,0}$ is a holomorphic line bundle over $\mathcal{C}$.

Note that for Teichmüller curves corresponding to the strata of Abelian differentials the splitting is always nontrivial, since it contains a flat subbundle of rank 2 such that the restriction of the pseudo-Hermitian form to this subbundle has signature $(1,1)$. The corresponding line bundle $\mathbb{L}^{1,0}$ is the tautological line bundle over the Teichmüller curve.

The curve $\mathcal{C}$ may have a finite number of cusps and conical points, so we need to consider the Deligne extension of the holomorphic line bundle $\mathbb{L}^{1,0}$, denoted by $\overline{\mathbb{L}^{1,0}}$: it becomes an orbifold vector bundle at the cusps and conical points. So it has an orbifold degree, which in general is not an integer, but a rational number.

Let $\chi(\mathcal{C})$ be the generalized Euler characteristic of $\mathcal{C}$: it is given by the formula

$$\chi(\mathcal{C})=2-2g-n_{\mathcal{C}}+\sum_i(k_i-1),$$

where $n_{\mathcal{C}}$ is the number of cusps on $\mathcal{C}$, and $2\pi k_i$ is the cone angle of the i-th conical point.

Theorem 1. *If $\chi(\mathcal{C})\geq 0$, then $\mathcal{C}$ is not a Teichmüller curve.*

Suppose that $\chi(\mathcal{C})<0$. For any flat subbundle $\mathbb{L}$ of the Hodge bundle over $\mathcal{C}$ satisfying the above assumptions, one has

$$\deg\overline{\mathbb{L}^{1,0}}\leq-\frac{\chi(\mathcal{C})}{2}\,. \tag{1.1}$$

If the equality is attained, then $\mathcal{C}$ is a Teichmüller curve, and the line bundle $\mathbb{L}^{1,0}$ is the tautological bundle.

Any Teichmüller curve corresponding to a stratum of Abelian differentials admits a flat subbundle $\mathbb{L}$ of the Hodge bundle satisfying the above conditions, such that

$$\deg\overline{\mathbb{L}^{1,0}}=-\frac{\chi(\mathcal{C})}{2}\,.$$

The first statement of the theorem results from the fact that any Teichmüller curve has negative curvature, or equivalently, an orbifold genus strictly greater than 1. So from now on, we assume that $\chi(\mathcal{C})<0$, that is, $\mathcal{C}$ is hyperbolic.

Remark 1. We have to admit that our criterion does not directly detect Teichmüller curves corresponding to the strata of quadratic differentials. However, the canonical double covering construction associates to every such Teichmüller curve $\mathcal{C}$ in $\mathcal{M}_g$ a Teichmüller curve $\mathcal{C}'$ in $\mathcal{M}_{g'}$ in the moduli space of curves of larger genus, such that the new Teichmüller curve already corresponds to some stratum of Abelian differentials, and thus, would be detected by our criterion. The new Teichmüller curve $\mathcal{C}'$ is isomorphic to the initial curve $\mathcal{C}$.

Since any Riemann surface X' in the family $\mathcal{C}'$ admits a holomorphic involution which changes the sign of the corresponding Abelian differential, the Teichmüller curves corresponding to such a double covering construction can be identified. Thus, indirectly the criterion detects all Teichmüller curves.

Remark 2. As Martin Möller pointed out to the author, in the case $r = 2$ this theorem can be refound by algebraic methods. Inequality (1.1) is a specific case of Arakelov inequality (see e.g. [De, Lemme 3.2]), and the bound is attained if and only if $\mathbb{L}$ is maximal Higgs (see [Pe00] or [VZ] for generalization in higher dimension). So Möller's criterion applies here and gives the conclusion of the theorem.

1.3 Comparison of hyperbolic versus Teichmüller metric

Recall that at any point $X \in \mathcal{M}_g$ the tangent space $T_X\mathcal{M}_g$ is identified with the space of essentially bounded Beltrami differentials, which is in duality with the space of integrable quadratic differentials on X by the pairing $\langle \mu, q\rangle = \int_X q\mu$. So the cotangent bundle $T^*\mathcal{C}$ of $\mathcal{C}$ can be viewed as a suborbifold of $\mathcal{Q}_g$, the moduli space of quadratic differentials. We will denote the total space of the cotangent bundle to the curve $\mathcal{C}$ by $\tilde{\mathcal{C}}$, and points of $\tilde{\mathcal{C}}$ by (X, q), where X is a Riemann surface and q is a quadratic differential on X. The pullback of $\mathbb{L}$ to $\tilde{\mathcal{C}}$ will also be denoted by $\mathbb{L}$. In the following, we will identify the cotangent bundle $\tilde{\mathcal{C}}$ with the tangent bundle by duality, so a quadratic differential will be seen as a tangent vector to $\mathcal{C}$.

We start by comparison of the two natural metrics on $\mathcal{C}$: the canonical hyperbolic metric given by Riemann's uniformization theorem, and the induced Teichmüller metric. Both of these metrics are, infinitesimally, Finsler metrics, so they define norms on each tangent space $T_X\mathcal{C}$ of $\mathcal{C}$.

Lemma 1. *Globally, on $\mathcal{C}$, the hyperbolic metric is larger than the induced Teichmüller metric, that is, the hyperbolic distance between any two points is larger than the Teichmüller distance.*

Infinitesimally, on each tangent space $T_X\mathcal{C}$, the unit ball for the norm associated to the hyperbolic metric is included in the unit ball for the norm associated to the induced Teichmüller metric.

Proof. The proof is based on the notion of Kobayashi metric (cf [Hu]). The canonical hyperbolic metric on $\mathcal{C}$ is by definition the Kobayashi metric on $\mathcal{C}$, and by Royden's theorem, the Teichmüller metric is the Kobayashi metric on $\mathcal{M}_g$. So the statement of the lemma results from the property of contraction of the (global or infinitesimal) Kobayashi metric for the inclusion $\mathcal{C} \hookrightarrow \mathcal{M}_g$. □

Now we apply this lemma to the cotangent bundle $\tilde{\mathcal{C}}$, using the identification $\tilde{\mathcal{C}} \simeq T\mathcal{C}$.

Corollary 1. *Let $\gamma(t)$ be a geodesic on $\mathcal{C}$ for the hyperbolic metric. Let us denote by $\gamma(\tau)$ the same curve parameterized by the arc length for the Teichmüller metric restricted to $\mathcal{C}$. The corresponding derivatives will be denoted by $\gamma' = \frac{\partial\gamma}{\partial t}$ and $\dot\gamma = \frac{\partial\gamma}{\partial\tau}$. Let v be an element of $\mathbb{L}$ at $(\gamma(0), \gamma'(0)) = (X, q/\|q\|_{hyp}) \in \tilde{\mathcal{C}} \subset \mathcal{Q}_g$. Then the Lie derivatives of the norm of v along γ satisfy*

$$\left|\mathcal{L}_{\gamma'(0)} \log\|v\|\right| \le \left|\mathcal{L}_{\dot\gamma(0)} \log\|v\|\right| \tag{1.2}$$

Proof. Note that $\gamma'(0)$ and $\dot\gamma(0)$ are tangent vectors to the same curve parametrized in two ways, at the same point, so they are colinear:

$$\gamma'(0) = \alpha\dot\gamma(0)\,.$$

Since $\gamma'(0)$ is unitary for the hyperbolic metric, and $\dot\gamma(0)$ unitary for the Teichmüller metric, by Lemma 1, $|\alpha| \le 1$. The conclusion follows by the chain rule. □

Note that Corollary 1 is valid for any choice of the norm in the Hodge bundle provided the norm varies smoothly with respect to a variation of a point in the base of the bundle. In the next section we pass to a very special *Hodge norm*.

1.4 Variation of the Hodge norm

The natural Hermitian form is positive-definite on $H^{1,0}(X,\mathbb{C})$, so it induces a norm:

$$\|h\|_{1,0}^2 = \frac{i}{2}\int_X h \wedge \overline{h}\,.$$

Similarly, the intersection form is negative-definite on $H^{0,1}$ so its opposite defines a norm $\|.\|_{0,1}$ on $H^{0,1}$. Note that for every $h \in \mathbb{L}^{1,0}$, we have $\|\overline{h}\|_{0,1} = \|h\|_{1,0}$. We define the *Hodge norm* on $H^1(X,\mathbb{C})$ by $\|v\| = \|h\|_{1,0} + \|a\|_{0,1}$, where h is the holomorphic part of v and a the anti-holomorphic part. This is the norm that we will consider on $\mathbb{L} = \mathbb{L}^{1,0} \oplus \mathbb{L}^{0,1}$, by restriction. From now on we consider only the Hodge norm.

The second lemma gives a uniform bound for the variation of the Hodge norm in the direction of the Teichmüller flow (for the definition of the Teichmüller flow, see e.g., [Fo1, Section 1]).

Lemma 2 (G. Forni)**.** *Let v be a non trivial element of the fiber $H^1(X,\mathbb{C})$ at $(X,q) \in \tilde{\mathcal{C}}$. Then the Lie derivative of the Hodge norm of v along the Teichmüller flow satisfies the following inequality :*

$$|\mathcal{L} \log \|v\|| \leq 1\,. \tag{1.3}$$

Moreover, it is an equality if and only if $q = \omega^2$ with $\omega \in H^{1,0}(\mathbb{C})$ and

$$v \in Span_{\mathbb{C}}(\omega) \oplus Span_{\mathbb{C}}(\bar{\omega}) - \{0\}$$

where $Span_{\mathbb{C}}(\omega)$ is the tautological bundle.

The statement is the extension to the complex case of a lemma of G. Forni (see [Fo1, Lemma 2.1']), reformulated in [FMZ1, Cor. 2.1]. The original statement holds in $H^1(\mathbb{R})$ (with the Hodge norm), and by the Hodge representation theorem, it also holds in $H^{1,0}$, endowed with the norm $\|.\|_{1,0}$. By conjugation, we obtain the result in $H^{0,1}$ with the norm $\|.\|_{0,1}$. Finally, note that inequality (1.3) is equivalent to the following:

$$|\mathcal{L}\|v\|| \leq \|v\|.$$

So, applying this majoration to each component (holomorphic and antiholomorphic) of an element v of $H^1(\mathbb{C})$, endowed with the chosen norm $\|.\|$, we obtain the following inequalities:

$$|\mathcal{L}\|v\|| = |\mathcal{L}\|h\|_{1,0} + \mathcal{L}\|a\|_{0,1}| \leq |\mathcal{L}\|h\|_{1,0}| + |\mathcal{L}\|a\|_{0,1}| \leq \|h\|_{1,0} + \|a\|_{0,1} = \|v\|,$$

so the result holds in $H^1(\mathbb{C})$.

Note that there is another proof of inequality (1.3) in [Mö, Lemma 6.10], in terms of curvature of the metrics.

With this two lemmas we can achieve the proof of the theorem.

1.5 Criterion in terms of Lyapunov exponents

In this section we give an alternative version of the criterion, in terms of Lyapunov exponents. This version does not require any assumption on the signature of the pseudo-Hermitian form on the flat bundle $\mathbb{L}$, so it is more general, but it has less interest in practice, because Lyapunov exponents are harder to compute than orbifold degree.

Proposition 3. *Let $\mathcal{C}$ be a curve in the moduli space $\mathcal{M}_g$, with $\chi(\mathcal{C}) < 0$, endowed with a flat subbundle $\mathbb{L}$ of rank $r \geq 2$ of the complex Hodge bundle, equivariant for the Gauss-Manin connection. Consider the Lyapunov exponents associated to the parallel transport of fibers of $\mathbb{L}$ along the geodesic flow given by the hyperbolic metric on $\mathcal{C}$. The absolute values of all these Lyapunov exponents are bounded above by 1. If the bound is achieved, the curve $\mathcal{C}$ is a Teichmüller curve.*

Proof. Let us first explain where these Lyapunov exponents come from. Recall that $\mathcal{C}$ is endowed with a canonical hyperbolic metric, which gives us a geodesic flow g_t^{hyp} on $T_1\mathcal{C}$, the unit tangent bundle of $\mathcal{C}$, and by duality, on the unit cotangent bundle $\tilde{\mathcal{C}}^{(1)}$. We look at the parallel transport of fibers of $\mathbb{L}$, endowed with the Gauss-Manin connection, along this geodesic flow.

Let ν be the Liouville measure on $\tilde{\mathcal{C}}^{(1)}$.

Since $\mathbb{L}$ inherits a variation of the Hodge structure from the Hodge bundle, it has quasi-unipotent monodromy around any cusp (see [Sc, Th. 6.1]). So there exists a finite unramified cover $\hat{\mathcal{C}}$ of $\mathcal{C}$, such that the pullback of $\mathbb{L}$ on $\hat{\mathcal{C}}$ has unipotent monodromy around any cusp of $\hat{\mathcal{C}}$. Passing to this finite cover preserves the ergodicity of the geodesic flow that we consider on $\hat{\mathcal{C}}$ (because of the hyperbolic features of the geodesic flow and *Hopf's argument*, see e.g., Wilkinson's article [Wi]), and does not change the Lyapunov exponents. Actually, all results we will obtain on $\mathcal{C}$ lift to $\hat{\mathcal{C}}$, so up to passing to this cover, we will assume in the rest of this paper that the monodromy of $\mathbb{L}$ is unipotent around any cusp of $\mathcal{C}$.

So with this assumption and thanks to the majoration given by Corollary 1 and Lemma 2, the cocyle associated to the geodesic flow is log-integrable. The Oseledets theorem (see [Os]) can be applied to this complex cocycle. We denote by λ_i the Lyapunov exponents and E_{λ_i} the corresponding subspaces.

By definition every vector v in $E_{\lambda_i}(q)$ expands with the rate :

$$\lambda_i = \lim_{t\to\infty} \frac{1}{t} \log \|v(\gamma_q(t))\|,$$

where γ_q is the geodesic for the hyperbolic metric starting at point X in the direction q.

We can write :

$$\lambda_i = \lim_{T\to\infty} \frac{1}{T} \int_0^T \frac{d}{dt} \log \|v(\gamma_q(t))\| dt\,. \tag{1.4}$$

So we have the following majoration :

$$\lambda_i \leq \lim_{T\to\infty} \frac{1}{T} \int_0^T \max_{v\in\mathbb{L}} \left(\frac{d}{dt} \log \|v(\gamma_q(t))\| \right) dt\,. \tag{1.5}$$

By Birkhoff's theorem, we have :

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \max_{v\in\mathbb{L}} \left(\frac{d}{dt} \log \|v(\gamma_q(t))\| \right) dt = \int_{\tilde{\mathcal{C}}^{(1)}} \max_{v\in\mathbb{L}} \left(\mathcal{L}_{\gamma_q'(0)} \log \|v(q)\| \right) d\nu(q). \tag{1.6}$$

It results from Corollary 1 and Lemma 2 that

$$\max_{v\in\mathbb{L}} \left| \mathcal{L}_{\gamma_q'(0)} \log \|v(q)\| \right| \leq 1, \tag{1.7}$$

so

$$\int_{\tilde{\mathcal{C}}^{(1)}} \max_{v\in\mathbb{L}} \left(\mathcal{L}_{\gamma_q'(0)} \log \|v(q)\| \right) d\nu(q) \leq 1. \tag{1.8}$$

Similary, we have:

$$\lambda_i \geq \int_{\tilde{\mathcal{C}}^{(1)}} \min_{v\in\mathbb{L}} \left(\mathcal{L}_{\gamma_q'(0)} \log \|v(q)\| \right) d\nu(q) \geq -1. \tag{1.9}$$

So we obtain that $|\lambda_i|$ is bounded by 1. Assume now that the bound is achieved for some index i. By symmetry of the spectrum (see [FMZ2, Theorem 4]), the exponent $-\lambda_i$ lies in the spectrum, so we can assume that $\lambda_i = 1$. It means that inequalities (1.5) and (1.8) are in fact equalities. Since the measure ν is normalized, and the modulus of the integrand in (1.8) is 1 at most (cf. (1.7)), it is almost everywhere equal to 1 and hence, by continuity, everywhere equal to 1. It means that inequalities of Corollary 1 and Lemma 2 are equalities.

The first equality case in Corollary 1 implies that the two metrics (hyperbolic and Teichmüller) coincide on $\mathcal{C}$. Hence $\mathcal{C}$ is invariant by the Teichmüller flow, so it is a Teichmüller curve.

Let us denote by v_1 the element of $\mathbb{L}$ at point (X, q) which maximizes the quantity $\mathcal{L} \log \|v\| \in [-1, 1]$. Similary, considering inequation (1.9) with $\lambda_i = -1$ gives an element v_2 which minimizes the same quantity. Clearly, v_1 and v_2 are independant. By Lemma 2, we have $q = \omega^2$ and $\mathrm{Span}_{\mathbb{C}}(v_1, v_2) = \mathrm{Span}_{\mathbb{C}}(\omega, \overline{\omega})$. So we obtain the additional information that $\mathbb{L}$ contains the complex tautological bundle. In particular, $\mathbb{L}^{1,0}$ corresponds to $\mathrm{Span}_{\mathbb{C}}(\omega)$. ☐

1.6 End of the proof

We will finish the proof of the theorem using that, with the additional assumption on the signature, there is only one non-negative Lyapunov exponent, which can be written in terms of degree of the line bundle $\mathbb{L}^{1,0}$. So we will be able to conlude with the previous proposition.

Since the pseudo-Hermitian form has signature $(1, r-1)$ on $\mathbb{L}$, there is at most one positive Lyapunov exponent, denoted by λ_1 (see [FMZ2, Theorem 4]).

As the monodromy is unipotent around cusps, the degree of the extended line bundle $\overline{\mathbb{L}^{1,0}}$ is the integral on $\mathcal{C}$ of the curvature form α (first Chern class), cf [Pe84, Prop. 3.4]. Then one has:

$$\int_{\mathcal{C}} \alpha = \deg \overline{\mathbb{L}^{1,0}}.$$

We use now the formula for the sum of the Lyapunov exponents of an invariant subbundle with respect to a geodesic flow defined by the hyperbolic metric on the curve $\mathcal{C}$. This formula was outlined by M. Kontsevich in [Ko2], developed by G. Forni in [Fo1]. I. Bouw and M. Möller suggested in [BM] an algebro-geometric interpretation of the numerator as the orbifold degree of the associate line bundle. As it was mentioned in [EKZ1, Section 2.3], the result holds for any abstract geodesic flow. So here we apply this result for the geodesic flow given by the hyperbolic metric on $\mathcal{C}$:

$$\lambda_1 = -\frac{2\int_{\mathcal{C}} \alpha}{\chi(\mathcal{C})} = -\frac{2 \deg \overline{\mathbb{L}^{1,0}}}{\chi(\mathcal{C})}. \tag{1.10}$$

This equality together with the previous proposition prove the second part of the theorem. The last statement of the theorem underlines the fact that any Teichmüller curve corresponding to a stratum of Abelian differentials admits a tautological bundle, which Lyapunov exponent is equal to 1.

This achieves the proof of the theorem.

Chapter 2

Geometry of periodic regions on flat surfaces and associated Siegel–Veech constants

This chapter will appear in
Geometriae Dedicata
under the title:
"Geometry of periodic regions on flat surfaces and associated Siegel–Veech constants".
It was written in collaboration with Max Bauer.

2.1 Introduction

2.1.1 Statement of some known results

Suppose that $M_{g,m}$ is a closed connected oriented surface M_g of genus g with a set of m labelled marked points $\Sigma = \{P_1, \ldots, P_m\}$.

By a *translation surface* S we mean a flat Riemannian metric and a parallel vector field on $M_{g,m}$. The metric has cone type singularities at all of the points P_i of Σ where the total angle is of the form $2\pi(d_i + 1)$ for some integer $d_i \geq 1$.

Each geodesic on a translation surface moves in a constant direction, so geodesics do not have self intersections and a regular geodesic that connects a non-singular point to itself comes back with the same angle, so it is a periodic geodesic. A periodic geodesic is always part of a maximal connected periodic region: a maximal cylinder of parallel periodic geodesics of the same length. We refer to such a maximal cylinder as a *periodic cylinder* or, for short, a *cylinder*, and we say that the common length of the periodic geodesics that make up the cylinder is the *width* of the cylinder. The number $N_{cyl}(S, \mathcal{C}, L)$ of cylinders of width less than L grows like $c\pi L^2$: a first fundamental result of Masur [Ma2, Ma3] states that there exist two positive constants c_1 and c_2 such that $c_1\pi L^2 \leq N_{cyl}(S, \mathcal{C}, L) \leq c_2\pi L^2$. Using this fact, Eskin and Masur [EMa] prove the deep theorem that for almost every surface in a stratum there is an exact asymptotics $N_{cyl}(S, \mathcal{C}, L) \sim c\pi L^2$. The constant c is called a *Siegel–Veech constant*. The main tool to study Siegel–Veech constants is the method of Veech [Ve3] that also showed the exact asymptotics in a more general setting, but for a weaker form of convergence.

The main object of this paper is the computation of various Siegel–Veech constants related to counting periodic regions in different ways. Siegel–Veech constants are particularly interesting because they appear in various fields. In arithmetic, they give the asymptotics of the number of primitive points in certain lattices. In dynamics they are related to the sum of Lyapunov exponents for the Teichmüller geodesic flow in any invariant suborbifold of a stratum of Abelian differentials by the main formula of [EKZ2]. For a Teichmüller curve, they have an algebro-geometric interpretation involving degrees of certain line bundles given by a formula of Bouw-Möller [BM]. Furthermore they are related to slopes of effective divisors and intersection theory in the moduli space of curves: this aspect is studied by Chen (see [Ch1, Ch2] for example) and Chen-Möller (see [CM] for example).

By identifying $\mathbb{R}^2$ and $\mathbb{C}$, a translation surface inherits from $\mathbb{C}$ a complex structure on M_g and a holomorphic one form (*Abelian differential*) ω. A zero of ω of order d corresponds to a conical singularity of angle $2\pi(d+1)$. There is a one to one correspondence between translation surfaces endowed with a distinguished direction and Abelian differentials. If we denote by $\alpha = (d_1, \ldots, d_m)$ the orders of the zeros of ω then we have $\sum d_i = 2g - 2$.

For a given $\alpha = (d_1, \ldots, d_m)$ such that $\sum d_i = 2g - 2$ and $d_i \geq 1$, for $i = 1, \ldots, m$, we consider the stratum $\mathcal{H}(\alpha)$ of the moduli space of Abelian differentials on $M_{g,m}$ that have zeros at the points of Σ of orders $(d_1, \ldots, d_m)$, or equivalently the moduli space of translation surfaces S with singularities at the points of Σ of angles $(2\pi(d_1+1), \ldots, 2\pi(d_m+1))$. The dimension of $\mathcal{H}(\alpha)$ is $\dim_{\mathbb{C}} \mathcal{H}(\alpha) = 2g + m - 1$. The stratum $\mathcal{H}(\alpha)$ may be non connected [Ve2] but contains at most three connected components [KZ]. The stratum $\mathcal{H}(\alpha)$ admits a volume element [Ma1, Ve1] that induces a finite $SL(2,\mathbb{R})$-invariant measure on the hyperspace $\mathcal{H}_1(\alpha)$ of area one surfaces in $\mathcal{H}(\alpha)$. The volumes of the connected components of the strata of Abelian differentials were effectively computed by A. Eskin and A. Okounkov [EOk1].

The boundary of a periodic cylinder contains singularities. Generically, each boundary component contains exactly one singularity so it is a *closed saddle connection*, i.e. a geodesic that joins a singularity to itself (and contains no other singularity).

Consider a translation surface S. For each positive real number $L > 0$, denote by $N_{\text{cyl}}(S, L)$ the number of periodic cylinders in S of width at most L and by $N_{\text{area}}(S, L)$ the total area of these cylinders. It was shown in [EMa] that

Theorem (Eskin–Masur)**.** *Let $\mathcal{H}(\alpha)$ be a stratum, where $\alpha = (d_1, \ldots, d_m)$ with $d_i \geq 1$, for $i = 1, \ldots, m$. For every connected component K of $\mathcal{H}_1(\alpha)$, there exist constants $c_{\text{cyl}}(K)$ and $c_{\text{area}}(K)$ such that for almost every translation surface S in K one has*

$$\lim_{L\to\infty} \frac{N_{\text{cyl}}(S, L)}{\pi L^2} = c_{\text{cyl}}(K) \qquad \lim_{L\to\infty} \frac{N_{\text{area}}(S, L)}{\pi L^2} = c_{\text{area}}(K).$$

The *Siegel–Veech constants* $c_{\text{cyl}}(K)$ and $c_{\text{area}}(K)$ only depend on K. An earlier version of this result (in a more general setting) where convergence is replaced by convergence in L^1 was proved in [Ve3].

The results in [EMa, Ve3] assure the existence of quadratic asymptotics (Siegel–Veech constants) if one counts cylinders with weights (under certain conditions). The existence of all the Siegel–Veech constants we consider in this paper is justified by these results (see section 2.1.3). The function $N_{area}(S, L)$ for example counts cylinders with weight the area of the cylinder.

One can also only count cylinders with sufficiently big area: for $p \in [0, 1)$, denote by $N_{\text{cyl}, A\geq p}(S, L)$ the number of cylinders in S of width at most L and of area at least p. We denote the corresponding Siegel–Veech constant by $c_{\text{cyl}, A\geq p}(K)$. It is shown in [Vo] that

Theorem (Vorobets [Vo])**.** *Let $\mathcal{H}(\alpha)$ be a stratum, where $\alpha = (d_1, \ldots, d_m)$ with $d_i \geq 1$, for $i = 1, \ldots, m$. Then for any connected component K of $\mathcal{H}_1(\alpha)$*

$$\text{(a)} \quad c_{\text{mean area}}(K) = \frac{c_{\text{area}}(K)}{c_{\text{cyl}}(K)} = \frac{1}{2g + m - 2} = \frac{1}{\dim_{\mathbb{C}} \mathcal{H}(\alpha) - 1}$$

$$\text{(b)} \quad \frac{c_{\text{cyl}, A\geq p}(K)}{c_{\text{cyl}}(K)} = (1-p)^{2g+m-3} = (1-p)^{\dim_{\mathbb{C}} \mathcal{H}(\alpha) - 2}.$$

The ratio $c_{\text{mean area}}(K)$ of part a) of the theorem can be interpreted as the (asymptotic) mean area of a cylinder on a generic surface in K in the following sense: for almost any surface M in K one has $\dfrac{c_{\text{area}}(K)}{c_{\text{cyl}}(K)} = \lim\limits_{L\to\infty} \dfrac{N_{\text{area}}(M, L)}{N_{\text{cyl}}(M, L)}$.

Part b) of the theorem is an answer to a question of Veech in [Ve3] where the autor asks if there is a simple formula for $\dfrac{c_{\text{cyl}, A\geq p}(K)}{c_{\text{cyl}, A\geq 0}(K)}$ (which is the same as $\dfrac{c_{\text{cyl}, A\geq p}(K)}{c_{cyl}(K)}$).

As a by-product of our results using the methods from [EMZ] we get an alternative proof of the theorem of Vorobets by evaluating an explicit integral that is a simplified version of the integral used in [EMZ].

The methods from [EMZ] allow for the computation of various other Siegel–Veech constants associated to counting cylinders for more specific data that we allude to next.

It might happen that the geodesic flow in a given direction on M contains several (maximal) periodic cylinders in that direction. Generically, this only happens if the boundary saddle connections of the cylinders are homologous. The boundary saddle connections might be part of a larger family of homologous saddle connections, where the extra saddle connections do not bound a cylinder. We refer to such a family as a *configuration of homologous saddle connections* or simply as a *configuration*. A topological representation of such configurations can be obtained by taking blocks of surfaces as in figures 2.6, 2.7, and 2.8 (the surfaces are drawn as tori, but might have arbitrary genus), arranging them in a cyclic order and then identifying the boundary components.

Note that the fact that the saddle connections are homologous implies that they are all of the same length. Call this the *length* of the configuration. It also implies that a configuration persists under small deformations of the translation structure.

One says that two configurations of homologous saddle connections correspond to the same *topological type* if the saddle connections are based at the same singularities, if the complementary regions are of the same topological type and have the same number and type of singularities, e.t.c. In particular, they always have the same number of complementary cylinders. The length of the configuration is the common width of the cylinders coming from the configuration.

For each connected component K of a stratum there are only a finite number of *admissible* topological types of configurations, i.e. topological types of configurations that are realized on at least one surface in K. In fact almost all surfaces share the same admissible topological types. We will consider only configurations of this special type. Their distinguishing feature is that they persist under any small deformation. For related counting problems in the case of general periodic components (so not configurations), see [Na, Li].

For S in K, denote by $N_{\mathrm{conf}}(S, \mathcal{C}, L)$ the number of configurations of homologous saddle connections on S of type $\mathcal{C}$ and whose length is at most L.

Theorem (Eskin–Masur–Zorich, [EMZ]). *For a given connected component K of a stratum $\mathcal{H}_1(\alpha)$ and an admissible topological type $\mathcal{C}$ of configurations there exists a Siegel–Veech constant $c_{\mathrm{conf}}(K, \mathcal{C})$ such that for almost any surface S in K one has*

$$\lim_{L\to\infty} \frac{N_{\mathrm{conf}}(S, \mathcal{C}, L)}{\pi L^2} = c_{\mathrm{conf}}(K, \mathcal{C}).$$

The authors of [EMZ] give a method to compute these Siegel–Veech constants.

2.1.2 Statement of results

We denote by $N_{\mathrm{cyl}}(S, \mathcal{C}, L)$ the number of cylinders of length less than L coming from a configuration of type $\mathcal{C}$. For a real number $p \geq 0$ we denote by $N_{\mathrm{area}^p}(S, \mathcal{C}, L)$ the sum of the p-th power of the area of each of these cylinders. We denote the corresponding Siegel–Veech constants by $c_{\mathrm{cyl}}(K, \mathcal{C})$, resp. $c_{\mathrm{area}^p}(K, \mathcal{C})$. For $p = 1$ we write $c_{\mathrm{area}}(K, \mathcal{C})$. Note that if $\mathcal{C}$ comes with q cylinders then $c_{\mathrm{cyl}}(K, \mathcal{C}) = q c_{\mathrm{conf}}(K, \mathcal{C})$.

It follows from the general result of [EMa] that there is a Siegel–Veech constant $c_{\mathrm{area}^p}(K, \mathcal{C})$ such that for almost any surface S in K one has

$$\lim_{L\to\infty} \frac{N_{\mathrm{area}^p}(S, \mathcal{C}, L)}{\pi L^2} = c_{\mathrm{area}^p}(K, \mathcal{C}).$$

The methods from [EMZ] can be applied to compute $c_{\mathrm{area}^p}(K, \mathcal{C})$ in a way similar to the computation of $c_{\mathrm{conf}}(K, \mathcal{C})$. The expression for $c_{\mathrm{area}^p}(K, \mathcal{C})$ contains a constant M that depends only on combinatorial data such as the dimension of the stratum, the order of the singularities and the possible symmetries. It is given by an explicit formula in § 13.3. of [EMZ]. It also contains the "principal boundary stratum" $\mathcal{H}_1(\alpha')$ determined by K and $\mathcal{C}$ (the stratum of possibly disconnected surfaces we get by contracting the closed saddle connections of the configuration). The constant

n always denotes the complex dimension of $\mathcal{H}_1(\alpha')$. If $\mathcal{C}$ comes with q cylinders then we have $n = \dim_{\mathbb{C}} \mathcal{H}(\alpha) - q - 1$.

With this notation we show in section 2.2.2,

Theorem 2. *Given a real number $p \geq 0$. Let $\mathcal{H}(\alpha)$ be a stratum, where $\alpha = (d_1, \dots, d_m)$ and $d_i \geq 1$, for $i = 1, \dots, m$. Let K be a connected component of $\mathcal{H}_1(\alpha)$ and $\mathcal{C}$ an admissible topological type of configuration containing $q \geq 1$ cylinders. Then*

$$c_{\mathrm{area}^p}(K, \mathcal{C}) = M \cdot \frac{\mathrm{Vol}(\mathcal{H}_1(\alpha'))}{\mathrm{Vol}(K)} \cdot \frac{(n-1)!}{(p+1) \cdot (p+2) \cdots (p+q+n-1)} \cdot q$$
$$c_{\mathrm{cyl}}(K, \mathcal{C}) = M \cdot \frac{\mathrm{Vol}(\mathcal{H}_1(\alpha'))}{\mathrm{Vol}(K)} \cdot \frac{(n-1)!}{(n+q-1)!} \cdot q$$
$$c_{area}(K, \mathcal{C}) = M \cdot \frac{\mathrm{Vol}(\mathcal{H}_1(\alpha'))}{\mathrm{Vol}(\mathcal{H}_1(\alpha))} \cdot \frac{(n-1)!}{(n+q)!} \cdot q$$

where $n = \dim_{\mathbb{C}} \mathcal{H}(\alpha) - q - 1$. M denotes the combinatorial constant given in § 13.3. of [EMZ] and $\mathcal{H}_1(\alpha')$ denotes the principal boundary stratum.

Note that $c_{\mathrm{area}}(K, \mathcal{C}) = c_{\mathrm{area}^1}(K, \mathcal{C})$ and $c_{\mathrm{cyl}}(K, \mathcal{C}) = c_{\mathrm{area}^0}(K, \mathcal{C})$, so the second and third equation of the previous theorem follow from the first.

Remark. a. Evaluation of c_{area^p} is motivated by the question of M. Möller related to the study of quasimodular properties of the related counting function.

b. We recall that the saddle connections in a configuration of given type $\mathcal{C}$ can be named. Choose and fix one of the saddle connections that bounds a cylinder. In the proof of theorem 2 we show that if we only consider the area of this cylinder then we get the same formulas for $c_{\mathrm{area}^p}(K, \mathcal{C})$ and $c_{\mathrm{area}}(K, \mathcal{C})$ except that the factor q is missing.

We get as a corollary:

Corollary 1. *Given a real number $p \geq 0$ and let $\mathcal{H}(\alpha)$ be a stratum of Abelian differentials on a surface $M_{g,m}$, where $\alpha = (d_1, \dots, d_m)$ with $d_i \geq 1$, for $i = 1, \dots, m$. Then for any connected component K of $\mathcal{H}_1(\alpha)$ and any admissible type $\mathcal{C}$ of configuration containing at least one cylinder,*

$$c_{\mathrm{mean\ area}^p}(K, \mathcal{C}) = \frac{c_{\mathrm{area}^p}(K, \mathcal{C})}{c_{\mathrm{cyl}}(K, \mathcal{C})} = \frac{(d-2)!}{(p+1) \cdot (p+2) \cdots (p+d-2)},$$

where $d = \dim_{\mathbb{C}} \mathcal{H}(\alpha) = 2g + m - 1$.

Remark. (a) The quotient of the corollary can be interpreted as the (asymptotic) mean area of a cylinder coming from a configuration of type $\mathcal{C}$, where the area is counted with a power p.

(b) For a natural number $p \geq 1$ we obtain

$$c_{\mathrm{mean\ area}^p}(K, \mathcal{C}) = \frac{1}{\binom{p+d-2}{p}}.$$

Define $N_{\mathrm{area}^p}(S, L)$ in the same way as we defined $N_{\mathrm{area}}(S, L)$ in section 2.1.1, except that the area of each cylinders is counted with a power p. If $c_{\mathrm{area}^p}(K)$ denotes the corresponding Siegel-Veech constant, then we have

$$c_{\mathrm{cyl}}(K) = \sum_{\mathcal{C}} c_{\mathrm{cyl}}(K, \mathcal{C}) \quad \text{and} \quad c_{\mathrm{area}^p}(K) = \sum_{\mathcal{C}} c_{\mathrm{area}^p}(K, \mathcal{C}),$$

where the sum is taken over all admissible topological types of configurations for K with at least one cylinder. This implies

Corollary 2. *Given a real number $p \geq 0$ and let $\mathcal{H}(\alpha)$ be a stratum, where $\alpha = (d_1, \dots, d_m)$ with $d_i \geq 1$, for $i = 1, \dots, m$. Then for any connected component K of $\mathcal{H}_1(\alpha)$*

$$c_{\mathrm{mean\ area}^p}(K) = \frac{c_{\mathrm{area}^p}(K)}{c_{\mathrm{cyl}}(K)} = \frac{(d-2)!}{(p+1) \cdot (p+2) \cdots (p+d-2)},$$

where $d = \dim_{\mathbb{C}} \mathcal{H}(\alpha) = 2g + m - 1$.

For $p = 1$, corollary 2 becomes part (a) of the theorem of Vorobets as stated in section 2.1.1. Corollary 1 gives more detailed information as corollary 2. For example, the mean area $c_{\text{mean area}^p}(K, \mathcal{C})$ of a cylinder coming from a configuration of type $\mathcal{C}$ does not depend on the number of cylinders. This means in particular that the mean area of a cylinder is the same if the cylinder makes up the whole periodic region of a configuration or if the periodic region is made up of several cylinders.

As a variation of the above, we denote by $N_{\text{area}^p,\text{conf}}(S, \mathcal{C}, L)$ the p-th power of the total area of the periodic region (union of the cylinders) on S coming from a configuration of topological type $\mathcal{C}$ whose length is at most L. The corresponding Siegel-Veech constant is denoted by $c_{\text{area}^p,\text{conf}}(K, \mathcal{C})$. We show in section 2.2.3

Theorem 3. *Given a real number $p \geq 0$ and let $\mathcal{H}(\alpha)$ be a stratum, where $\alpha = (d_1, \ldots, d_m)$ and $d_i \geq 1$, for $i = 1, \ldots, m$. Let K be a connected component of $\mathcal{H}_1(\alpha)$ and $\mathcal{C}$ an admissible topological type of configuration containing $q \geq 1$ cylinders. Then*

$$\text{(a)} \quad c_{\text{area}^p,\text{conf}}(K, \mathcal{C}) = M \cdot \frac{\operatorname{Vol}(\mathcal{H}_1(\alpha'))}{\operatorname{Vol}(K)} \cdot \frac{(n-1)!}{(q-1)!} \frac{1}{(p+q)\cdots(p+q+n-1)}$$

$$\text{(b)} \quad \frac{c_{\text{area}^p,\text{conf}}(K, \mathcal{C})}{c_{\text{conf}}(K, \mathcal{C})} = \frac{q(q+1)\cdots(q+n-1)}{(p+q)(p+q+1)\cdots(p+q+n-1)}.$$

M denotes the combinatorial constant given in § 13.3. of [EMZ] and $\mathcal{H}_1(\alpha')$ denotes the principal boundary stratum.

Remark. (a) For a natural number $p \geq 1$ we obtain

$$\frac{c_{\text{area}^p,\text{conf}}(K, \mathcal{C})}{c_{\text{conf}}(K, \mathcal{C})} = \frac{q \cdot (q+1)\cdots(q+p-1)}{(d-1)\cdot d \cdots (d+p-2)},$$

where $d = \dim_{\mathbb{C}} \mathcal{H}(\alpha) - 1$.

(b) The quotient in part (b) of the preceeding theorem can be interpreted as the asymptotic mean area of the periodic part (taking the p-th power of the area). For $p = 1$ we get $qc_{\text{mean area}}(K, \mathcal{C})$, which is consistent, as $c_{\text{mean area}}(K, \mathcal{C})$ is the mean area of a cylinder.

(c) For $q = 1$ we have $c_{\text{area}^p,conf}(K, \mathcal{C}) = c_{\text{area}^p}(K, \mathcal{C})$.

We next count configurations with cylinders of large area. We recall that the saddle connections in a configuration of given type $\mathcal{C}$ can be named. Choose and fix one of the saddle connections that bound a cylinder and call this the first cylinder. Given $p \in [0, 1)$. Denote by $N_{\text{conf},A_1 \geq p}(S, \mathcal{C}, L)$ the number of configurations of type $\mathcal{C}$ of length at most L and such that the area of the first cylinder is at least p (of the area one surface S). We denote by $c_{\text{conf},A_1 \geq p}(K, \mathcal{C})$ the corresponding Siegel–Veech constant. Note that we have $c_{\text{conf},A_1 \geq 0}(K, \mathcal{C}) = c_{\text{conf}}(K, \mathcal{C})$. We show in section 2.2.5

Theorem 4. *Given $p \in [0, 1)$. Let $\mathcal{H}(\alpha)$ be a stratum of Abelian differentials on a surface $M_{g,m}$, where $\alpha = (d_1, \ldots, d_m)$ with $d_i \geq 1$, for $i = 1, \ldots, m$. Then for any connected component K of $\mathcal{H}_1(\alpha)$ and any admissible topological type $\mathcal{C}$ of configuration for K containing at least one cylinder,*

$$\frac{c_{\text{conf},A_1 \geq p}(K, \mathcal{C})}{c_{\text{conf}}(K, \mathcal{C})} = (1-p)^{2g+m-3} = (1-p)^{\dim_{\mathbb{C}} \mathcal{H}(\alpha) - 2}.$$

Summing over all configurations we get as a corollary part (b) of the theorem of Vorobets but our result contains more detailed information.

We next count configurations with periodic regions of large area. Let $\mathcal{C}$ be a topological type of configuration that comes with $q \geq 1$ cylinders. For $p \in [0, 1)$, denote by $N_{\text{conf},A \geq p}(S, \mathcal{C}, L)$ the number of configurations on S of type $\mathcal{C}$ of length at most L and such that the total area of the q cylinders is at least p (of the area one surface S). We denote the corresponding Siegel–Veech constant by $c_{\text{conf},A \geq p}(K, \mathcal{C})$.

The *incomplete Beta function* $B(x; a, b)$ and the *regularized incomplete Beta function* $\mathrm{I}(x; a, b)$ are defined for $x \in [0, 1]$ by

$$B(x; a, b) = \int_0^x u^{a-1}(1-u)^{b-1} \mathrm{d}u, \qquad \mathrm{I}(x; a, b) = \frac{B(x; a, b)}{B(1; a, b)} = \frac{B(x; a, b)}{B(a, b)}.$$

For more details about the incomplete Beta function see section 2.4: Figure 2.9 represents the density function $\frac{d}{dx}\,\mathrm{I}(x;a,b)$ for various values for a and b.

With this notation we show in section 2.2.4:

Theorem 5. *Given $p \in [0,1)$, and let $\mathcal{H}(\alpha)$ be a stratum, where $\alpha = (d_1,\ldots,d_m)$ with $d_i \geq 1$, for $i = 1,\ldots,m$. Suppose that K is a connected component of $\mathcal{H}_1(\alpha)$ and that $\mathcal{C}$ is an admissible topological type of configurations for K containing exactly q cylinders.*

$$\frac{c_{\mathrm{conf},A\geq p}(K,\mathcal{C})}{c_{\mathrm{conf}}(K,\mathcal{C})} = \mathrm{I}(1-p;n,q) = (1-p)^n \sum_{k=0}^{q-1} \binom{n-1+k}{k} p^k,$$

where $n = \dim_{\mathbb{C}} \mathcal{H}(\alpha) - q - 1 = \dim_{\mathbb{C}} \mathcal{H}(\alpha')$.

See figure 2.1 for the graph of $\mathrm{I}(1-p;n,q)$ for various values for n and q.

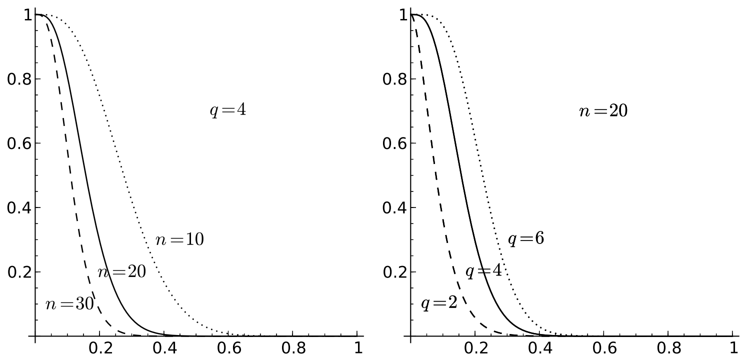


Figure 2.1: Graphs of the function $f(p) = \dfrac{c_{\mathrm{conf},A>p}(K,\mathcal{C})}{c_{\mathrm{conf}}(K,\mathcal{C})}$

Remark. a. The fraction $\dfrac{c_{\mathrm{conf},A\geq p}(K,\mathcal{C})}{c_{\mathrm{conf}}(K,\mathcal{C})}$ can be interpreted as the mean number of configurations of type $\mathcal{C}$ whose periodic complementary region is big, that is, the total area of the cylinders is at least p of the area of the surface.

b. $\mathrm{I}(1;n,q) = 1$ and

$$\lim_{p\to 1} \frac{I(0;n,q)}{(1-p)^n} = \sum_{=0}^{q-1} \binom{n+l-1}{l} = \binom{n+q-1}{n},$$

so

$$\frac{c_{\mathrm{conf},A\geq p}(K,\mathcal{C})}{c_{\mathrm{conf}}(K,\mathcal{C})} \sim (1-p)^n \binom{n+q-1}{n} \text{ as } p \to 1.$$

In this form we can compare the result with the previous one for one cylinder, given in Theorem 4.

In section 2.2.6 we consider the problem of correlation between the area of two cylinders. Let $\mathcal{C}$ be an admissible configuration for a connected component K that comes with at least two cylinders. Choose (and fix) two cylinders and let $p, p_1 \in [0,1)$. We denote by $N_{A_2\geq p, A_1\geq p_1}(S,\mathcal{C},L)$ the number of configurations of width length at least L such that the area A_1 of the first cylinder is at least p_1 and such that the area A_2 of the second cylinder is at least p of the remaining surface, i.e. it is at least $p(1-A_1)$. We denote by $c_{A_2\geq p, A_1\geq p_1}(K,\mathcal{C})$ the corresponding Siegel–Veech constant. To simplify notation we will write $c_{A_1\geq p_1}(K,\mathcal{C})$ instead of $c_{\mathrm{conf},A_1\geq p_1}(K,\mathcal{C})$. We show in section 2.2.6,

Theorem 6. *For any connected component K of a stratum $\mathcal{H}_1(\alpha)$, where $\alpha = (d_1,\ldots,d_m)$ with $d_i \geq 1$, for $i = 1,\ldots,m$, and any admissible topological type $\mathcal{C}$ of configurations containing at least two cylinders,*

$$\frac{c_{A_2\geq p, A_1\geq p_1}(K,\mathcal{C})}{c_{A_1\geq p_1}(K,\mathcal{C})} = (1-p)^{\dim_{\mathbb{C}} \mathcal{H}(\alpha) - 3}.$$

The result does not depend on p_1.

Morally, we compute the asymptotic probability that, among configurations whose first cylinder has area p_1, we have a second cylinder with area at least $p(1-p_1)$. Comparing Theorems 6 and 4 we see that this is the probability that, among all configurations, the area of the second cylinder is at least p, except that the parameter space has one fewer dimension. So, in some sense, except for the fact that the area of the first cylinder gives a restriction on the range for the area of the second cylinder, the area of the second cylinder is independent of the area of the first cylinder.

In the results presented above we studied individual configurations. In the remaining part of the paper we study extremal properties of configurations among all configurations in a given stratum or even among all strata for a fixed genus.

In section 2.3.1 we address the question of finding topological types of configurations $\mathcal{C}$ (admissible for some connected component K) that maximizes

$$c_{\text{mean area conf}}(K,\mathcal{C}) = \frac{c_{\text{area}}(K,\mathcal{C})}{c_{\text{conf}}(K,\mathcal{C})}.$$

The constant $c_{\text{mean area conf}}(K,\mathcal{C})$ can be interpreted as the asymptotic mean area of the periodic part (union of the cylinders) of the complementary region of a configuration of topological type $\mathcal{C}$.

Each stratum has at most three connected components that are classified by the invariants "hyperellipticity", and "parity of spin structure". (We recall in section 2.3.2 the classification of connected components from [KZ].)

The quantity $c_{\text{mean area conf}}(K,\mathcal{C})$ varies considerably among strata. For the connected stratum $\mathcal{H}(1,1,\dots,1)$, the maximal value of $c_{\text{mean area conf}}(K,\mathcal{C})$ over all the configurations is $\frac{1}{4}$. For the connected component $\mathcal{H}^{hyp}(g-1,g-1)$ it is equal to $\frac{1}{2g}$. The following proposition gives an uniform bound on the ratio $c_{\text{mean area conf}}(K,\mathcal{C})$.

Theorem 7. *Let K be any connected component of a stratum $\mathcal{H}(\alpha)$ and $\mathcal{C}$ be any admissible topological type of configuration for K, then the asymptotic mean area of the periodic complementary regions satisfies*

$$c_{\text{mean area conf}}(K,\mathcal{C}) \leq \frac{1}{3}.$$

The maximum is attained for any genus $g \geq 2$: for each $g \geq 2$ there is a topological type of configuration $\mathcal{C}_g$ that is admissible for the component $\mathcal{H}^{odd}(2,2,\dots,2)$ ($g-1$ zeros of order 2) of Abelian differentials on a surface of genus g such that the corresponding constant $c_{\text{mean area conf}}$ is $\frac{1}{3}$.

To prove this result we need to determine the maximal number of cylinders that can come from any configuration which is admissible for a fixed stratum $\mathcal{H}(\alpha)$. We insist on the fact that we compute here the number of cylinders in rigid collections of saddle connections, which is different from the studies of Naveh [Na] and Lindsey [Li] where they count the number of parallel cylinders.

In section 2.3.2 we answer a question of A. Eskin and A. Wright: is it possible to find in each connected component of each stratum a topological type of configuration whose complementary regions are tori with boundary and cylinders.

The answer depends on the connected component. We show (Proposition 5): for hyperelliptic components this is not possible; for the components with even spin structure when the genus is even this is not possible unless we allow one of the complementary regions to be a genus two surface; in all other cases this is possible.

2.1.3 Notation

We introduce here most of the notation we need for the computation of Siegel–Veech constants. For survey material on Abelian differentials and translation surfaces see [GJ, Zo4, MT]. For the exact definition of configurations and related results see [EMZ].

Let $M_{g,n}$ denote a closed oriented surface M_g of genus g on which there are m marked points $\Sigma = \{P_1,\dots,P_m\}$. By (R,ω) we denote a Riemann surface structure R on M_g together with an Abelian differential ω. If ω is not identically zero then Σ is the set of zeros of ω. We usually only write ω for (R,ω). If we denote by $(d_1,\dots,d_m)$ the orders of the zeros of ω then we have $\sum d_i = 2g-2$.

The form ω can be used to define an atlas of adapted coordinates on R. In these adapted coordinates the Abelian differential ω becomes dz in a neighborhood of any point of $R\setminus\Sigma$ and is $(d_i+1)w^{d_i}dw = d(w^{d_i+1})$ in a neighborhood of a point $P_i \in \Sigma$, where d_i is the order of the zero P_i. Transition functions away from the zeros for these adapted coordinates are translations. We refer to a surface together with an atlas whose transition functions are translations as *translation surfaces*.

Using such an atlas, $R\setminus\Sigma$ inherits from the complex plane a flat (zero curvature) Riemannian metric. A zero of order d_i of the Abelian differential (that is a regular point of the Riemann surface structure) corresponds to a conical singularity of the flat metric of total angle $2\pi(d_i+1)$. These points are also called *saddles*.

The horizontal unit vector field on $\mathbb{C}$ pulls back by adapted coordinates to a horizontal unit vector field on $R\setminus\Sigma$. Away from the singularities, the leaves of the corresponding foliation are geodesics with respect to the flat metric. In fact, for each $\theta\in[0,2\pi[$ we have a unit vector field and so a foliation in that direction that comes from the unit vector field in direction θ of the complex plane.

The converse construction is also possible: suppose that S is a translation surface structure on $M_{g,m}$, i.e. an atlas on $M_{g,m}$ whose transition functions are translations. A translation surface inherits from $\mathbb{R}^2$ a flat Riemannian metric and a parallel vector field on $M_{g,m}$. We assume that M_g is the metric completion of $M_{g,m}$. The points of Σ that are not regular points for the metric, are cone type singularities where the total angle is of the form $2\pi(d_i+1)$ for some $d_i\in\mathbb{N}$.

By identifying $\mathbb{R}^2$ and $\mathbb{C}$, a translation surface inherits from $\mathbb{C}$ a complex (Riemann surface) structure on M_g and a holomorphic one form (*abelian differential*) ω. The zeros of ω are contained in Σ.

Each translation surfaces can be represented by a polygon in the plane whose edges come in pairs of parallel sides of the same length. Identifying each pair by a translation one obtains a translation surface. The vertices give rise to singularities (or regular points if the total angle is 2π.)

We identify two translation surface structures if there is a bijection of the underlying topological surface that is in local coordinates a translation. We identify two Abelian differentials (for some complex structures) if they are biholomorphically equivalent. There is then a one to one correspondance between Abelian differentials and translation surfaces. We denote by $\mathcal{H}$ the moduli space of Abelian differentials ω or equivalently of translation surfaces S. The moduli space $\mathcal{H}$ is an algebraic variety.

Given $\alpha=(d_1,\dots,d_m)$ such that $2g-2=\sum_i d_i$. The set $\mathcal{H}(\alpha)$ of Abelian differentials that share the same zero structure α is called a stratum. The stratum $\mathcal{H}(\alpha)$ is an algebraic subvariety that admits a natural affine structure and a natural "Lebesgue" volume element induced by this affine structure [Ma1, Ve1]. The dimension of $\mathcal{H}(\alpha)$ is $\dim_{\mathbb{C}}\mathcal{H}(\alpha)=2g+m-1$.

The area of the translation surface S defined by the Abelian differential $\omega=\phi(z)dz$ is $\int_S|\phi(z)|^2\mathrm{d}x\mathrm{d}y$. We denote by $\mathcal{H}_1(\alpha)$ the hyperspace of $\mathcal{H}(\alpha)$ of area one surfaces. Masur [Ma1] and Veech [Ve1] showed that $\mathcal{H}_1(\alpha)$ with the measure induced by the measure on $\mathcal{H}(\alpha)$ is of finite volume.

There is a natural $\mathrm{SL}(2,\mathbb{R})$ action on $\mathcal{H}(\alpha)$. An element $g\in\mathrm{SL}(2,\mathbb{R})$ acts on local coordinates by postcomposition of g. The measure on $\mathcal{H}(\alpha)$ and $\mathcal{H}_1(\alpha)$ is $\mathrm{SL}(2,\mathbb{R})$ invariant. If we represent a translation surfaces by a polygon in $\mathbb{R}^2$, then the action of g is the usual action on $\mathbb{R}^2$.

Geodesic segments in the flat metric have constant angle with respect to the flat metric, so a geodesic segment γ can be represented by a holonomy vector $\mathrm{hol}(\gamma)$ in $\mathbb{R}^2$. The angle and length of the vector is given by the direction and length of the geodesic segment. We will often use γ both for the geodesic segment and for the holonomy vector.

As we said above, a closed geodesic is always contained in a maximal cylinder of closed geodesics and each boundary component of the cylinder is generically a closed saddle connection. As all of the geodesics in the cylinder are represented by the same vector we say that this vector represents the cylinder. Its length is the *width* of the cylinder.

Suppose that on a translation surface we find in some direction a *configuration of homologous saddle connections*, meaning a maximal family of homologous saddle connections. All of the saddle connections are parallel and of the same length, so they share the same holonomy vector. Its length is the *length* of the configuration.

A configuration defines the following data: the named singularities the saddle connections are based at; the topological type of the complementary regions; the knowledge of which saddle connection bounds which complementary region, so in particular the cyclic order of the complementary regions; the order of the singularities (if there are any) in the interior of each complementary region; the singularity structure on the boundary (the type of boundary and angles at the boundary singularities to be described below).

We say that two configurations are of the same *topological type* if they define the same data. Almost all surfaces in a given connected component K share the same topological types of configurations that can be realized on the surface. We talk about an *admissible topological type* for K.

Denote by $V(S,\mathcal{C})$ the (discrete) set of holonomy vectors associated to configurations on S of type $\mathcal{C}$. We are allowed to associate weights to the holonomy vectors. Denote by $B(L) \subset \mathbb{R}^2$ the disk of radius L centered at the origin and by $N(S,\mathcal{C},L)$ the cardinality of $V(S,\mathcal{C}) \cap B(L)$, where the elements of $V(S,\mathcal{C}) \cap B(L)$ are counted with their weights. So if we write an element of $V(S,\mathcal{C})$ as $(v, w(v))$ where $w(v)$ is the weight of the holonomy vector v, then $N(S,\mathcal{C},L) = \sum_{v \in V(S,\mathcal{C}) \cap B(L)} w(v)$.

By using appropriate weights on the holonomy vectors the counting function $N(S,\mathcal{C},L)$ becomes the counting functions $N_{\text{conf}}(S,\mathcal{C},L)$, $N_{\text{cyl}}(S,\mathcal{C},L)$, $N_{\text{area}}(S,\mathcal{C},L)$ e.t.c. introduced in section 2.1.2. If for example we count holonomy vectors associated to a configuration with weight one (resp. the number of cylinders, total area of the cylinders) then $N(S,\mathcal{C},L)$ equals $N_{\text{conf}}(S,\mathcal{C},L)$ (resp. $N_{\text{cyl}}(S,\mathcal{C},L)$, $N_{\text{area}}(S,\mathcal{C},L)$).

Given a connected component K of some stratum $\mathcal{H}_1(\alpha)$ and an admissible topological type $\mathcal{C}$ for K. It follows from [EMa] that the set of (weighted) holonomy vectors $V(S,\mathcal{C})$ and the associated counting functions $N(S,\mathcal{C},L)$ we consider in this paper verify the following conditions :

(A) for every $g \in SL(2,\mathbb{R})$, $V(gS,\mathcal{C}) = gV(S,\mathcal{C})$.

(B) for every $S \in K$ there exists a constant $c(S) > 0$ such that $N(S,L,\mathcal{C}) \le c(S)L^2$. The constant $c(S)$ can be chosen uniformly on compact sets of K.

(C) there exist constants $L > 0$ and $\varepsilon > 0$ such that $N(S,L,\mathcal{C})$ is $L^{1+\varepsilon}(K,\mu)$ as a function of S.

In fact, the authors of [EMa] show that the above conditions are verified for the set $V(S)$ of holonomy vectors of closed saddle connections on S where the weight associated to a holonomy vector is the number of saddle connections on S that share this vector. The weights we use in this paper are "invariant under $\mathrm{SL}(2,\mathbb{R})$", so the sets $V(S,\mathcal{C})$ we use verify condition (A). The number of configurations is bounded by the number of saddle connections, and as we use bounded weights, the counting functions $N(S,\mathcal{C},L)$ we consider also satisfy conditions (B) and (C).

For $f \in C_0^\infty(\mathbb{R}^2)$ one defines the function $\hat{f} : K \to \mathbb{R}$ by

$$\hat{f}(S) = \sum_{v \in V(S,\mathcal{C})} w(v) f(v).$$

The fact that the sets $V(S,\mathcal{C})$ and associated counting functions $N(S,\mathcal{C},L)$ we consider in this paper satisfy the conditions (A), (B), (C), implies, using [EMa], that:

Theorem ([EMa, Ve3]). *Let K be a connected component of some stratum $\mathcal{H}_1(\alpha)$ of Abelian differentials and $\mathcal{C}$ an admissible topological type of saddle connections for K. Then for any of the sets $V = V(S,\mathcal{C},L)$ and associated counting functions $N_V(S,\mathcal{C},L)$ we consider in this paper there is a constant $c_V(K,\mathcal{C})$ such that the following holds :*

(a) For almost any translation surface S in K,

$$\lim_{L \to \infty} \frac{N_V(S,\mathcal{C},L)}{\pi L^2} = c_V(K,\mathcal{C}).$$

(b) For any $f \in C_0^\infty(\mathbb{R}^2)$,

$$\frac{1}{\mathrm{vol}(K)}\int_K \hat{f}(S)\mathrm{d}vol(S) = c_V(K,\mathcal{C})\int_{\mathbb{R}^2} f(x,y)\mathrm{d}x\mathrm{d}y.$$

If the convergence in (a) is replaced by convergence in L^1 then this theorem follows from a more general theorem under similar hypotheses in [Ve3]. In that paper, (b) is proved for integrable functions f of compact support. We will only use (b) for the characteristic function on a disk.

The paper [EMZ] explains how this last theorem can be used to compute the Siegel–Veech constant $c_{\mathrm{conf}}(K,\mathcal{C})$. The same method can be used to compute the other Siegel–Veech constants we consider in this paper.

The strategy used in [EMZ] is as follows: if we apply (b) for the characteristic function f_ε of the disc $B(\varepsilon)$ of radius ε centered at the origin then the integral of the right hand side becomes $\pi\varepsilon^2$ and we have

$$\hat{f}_\varepsilon(S) = \sum_{v\in V(S,\mathcal{C})\cap B(L)} w(v) = N(S,\mathcal{C},\varepsilon).$$

So

$$c_V(K,\mathcal{C}) = \frac{1}{\mathrm{vol}(K)}\frac{1}{\pi\varepsilon^2}\int_K N_V(S,\mathcal{C},\varepsilon)\mathrm{d}\,\mathrm{vol}(S).$$

Denote by $K(\varepsilon,\mathcal{C})$ the subset of translation surfaces in K that contain at least one configuration of type $\mathcal{C}$ of length smaller than ε. So $N_V(S,\mathcal{C},\varepsilon)$ is zero outside $K(\varepsilon,\mathcal{C})$. Denote by $K^{\mathrm{thick}}(\varepsilon,\mathcal{C})$ the subset of $K(\varepsilon,\mathcal{C})$ of translation surfaces that contain exactly one configuration of type $\mathcal{C}$ of length smaller than ε but contain no other closed saddle connection of length smaller than ε. Using a result from [EMa] the autors of [EMZ] show that

$$\mathrm{vol}(K(\varepsilon,\mathcal{C})) = \mathrm{vol}(K^{\mathrm{thick}}(\varepsilon,\mathcal{C})) + o(\varepsilon^2).$$

So

$$\begin{aligned} c_V(K,\mathcal{C}) &= \frac{1}{\mathrm{vol}(K)}\frac{1}{\pi\varepsilon^2}\int_{K(\varepsilon,\mathcal{C})} N_V(S,\mathcal{C},\varepsilon)\mathrm{d}\,\mathrm{vol}(S) \\ &= \lim_{\varepsilon\to 0}\frac{1}{\mathrm{vol}(K)}\frac{1}{\pi\varepsilon^2}\int_{K^{\mathrm{thick}}(\varepsilon,\mathcal{C})} N_V(S,\mathcal{C},\varepsilon)\mathrm{d}\,\mathrm{vol}(S). \end{aligned} \tag{2.1}$$

Consider a translation surface S in $K^{\mathrm{thick}}(\varepsilon,\mathcal{C})$. On S we have a configuration of closed saddle connections of topological type $\mathcal{C}$ of length smaller than ε and no other short closed saddle connection. Cutting along the closed saddle connections we decompose S into several pieces. There will be some, say $p \geq 1$, surfaces $S_1,\dots,S_p$ with boundary and some, say $q \geq 0$, periodic cylinders $C_1,\dots,C_q$, all having the same width. The pieces are arranged in a cyclic order. The boundary of each S_i is made up of two closed saddle connections.

For each connected component S_i, by taking out the boundary and then taking the compactification we get a surface with either one boundary component, a *figure eight*, as in the right part of figure 2.2, or we get a surface with two boundary components, a *pair of holes*, as in the right part of figure 2.3.

In the first case, the figure eight boundary describes two interior sectors of the surface of angles $2\pi(a'+1)$ and $2\pi(a''+1)$, for some integers $a', a'' \geq 0$. (Figure 2.2 illustrates the case $a' = 1$ and $a'' = 0$.) By shrinking the figure eight boundary to a point we produce a singularity of order $a' + a''$ if $a' + a'' \geq 1$, or a regular point if $a' + a'' = 0$. (See the left part of figure 2.2.)

In the second case, each of the boundary components comes with a boundary singularity of angles $\pi(2b'+3)$ and $\pi(2b''+3)$ for some integers $b', b'' \geq 0$. (Figure 2.3 illustrates the case $b' = 1$ and $b'' = 0$.) By shrinking the two boundary components we produce singularities (or regular points) of orders b' and b''. (See the left part of figure 2.3.)

The type of boundary, figure eight or pair of holes, and the associated angles is what we referred to above as "singularity structure on the boundary".

We get in this way closed surfaces S'_i that belong to some stratum $\mathcal{H}(\alpha'_i)$, for $i = 1,\dots,p$. We write $\alpha' = \sqcup_{i=1}^p \alpha'_i$ and $\mathcal{H}(\alpha') = \Pi_{i=1}^p \mathcal{H}(\alpha'_i)$. We will say that the surface S' whose connected

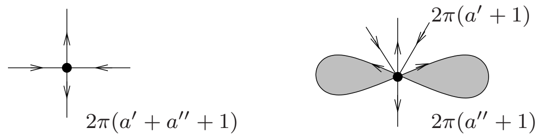


Figure 2.2: Figure eight construction

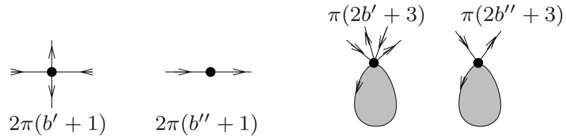


Figure 2.3: Creating a pair of holes

components are $S'_1, \ldots, S'_p$ belongs to $\mathcal{H}(\alpha')$. We say that S' belongs to the *principal boundary* of $\mathcal{H}(\alpha)$ determined by $\mathcal{C}$ and that $\mathcal{H}(\alpha')$ is the corresponding *principal boundary stratum*. All this data (topological type of complementary regions, cyclic order, boundary stratum, e.t.c.) are the same for any configuration of the same topological type $\mathcal{C}$. In fact the topological type is characterized by this data.

By shrinking a closed saddle connection, the singular point it is based at might become a regular point with total angle 2π. In this case the regular point will be considered as a marked point of order 0. So α'_i can contain one or two 0.

The procedure of shrinking saddle connections can be reversed. We summarize the description form [EMZ] in case where $\mathcal{H}(\alpha)$ has only one connected component K. Start with a (maybe disconnected) surface T' in $\mathcal{H}(\alpha')$ and call the connected components T'_i. Choose some holonomy vector γ in $B(\varepsilon)$. There are two types of surgery. A *figure eight surgery* where we start with a singularity or a marked point of order $a \geq 0$, choose $a', a'' \geq 0$ such that $a = a' + a''$ and then metrically create a figure eight boundary that consists of two saddle connections in direction γ that are of length $|\gamma|$ and are based at the same singularity. There will be two sectors of angles $2\pi(a'+1)$ and $2\pi(a''+1)$ (see figure 2.2). A *pair of holes surgery* where we start with two points that are either a singularity or a marked point of orders $b' \geq 0$ and $b'' \geq 0$ and then create a boundary saddle connection at each singularity. The pair of holes boundary has two boundary singularities of angles $\pi(2b'+3)$ and $\pi(2b''+3)$ (see figure 2.3).

By performing an appropriate surgery on each T'_i we obtain surfaces T_i that are homeomorphic to the surfaces S_i and have the same type of singularities in the interior and on the boundary. We then take q cylinders C_j with a marked point on each boundary. We finally combine the surfaces T_i and the cylinders C_j in the way prescribed by the topological type $\mathcal{C}$ to produce a surface T in $K^{\text{thick}}(\varepsilon, \mathcal{C})$. We do this by identifying pairs of boundary components by an isometry that identifies boundary singularities. The boundary components give rise to a configuration of closed saddle connections of topological type $\mathcal{C}$. In fact each surface in $K^{\text{thick}}(\varepsilon, \mathcal{C})$ can be produced in this way.

The parameters used to produce surfaces in $K^{\text{thick}}(\varepsilon, \mathcal{C})$ are the following:

- a maybe disconnected surface in $\mathcal{H}(\alpha')$.
- a holonomy vector γ in $B(\varepsilon)$.
- a combinatorial constant M that only depends on the configuration. There is a $M : 1$ correspondence between the surfaces in $K^{\text{thick}}(\varepsilon, \mathcal{C})$ and the surfaces in $\mathcal{H}_1(\alpha')$. This is mainly due to the facts that at a zero of order k there are $k+1$ sectors of angle 2π where we can produce a saddle connection in the direction of γ and to possible symmetries of the surface in $K^{\text{thick}}(\varepsilon, \mathcal{C})$ (see § 13.3. of [EMZ]).
- the heights h_i of the q cylinders C_i (the width is given by $|\gamma|$).

- for each cylinder C_i a twist parameter $t_i \in [0, |\gamma|)$ that describes the relative position of the marked points on the two boundary components.

Remark. If $\mathcal{H}(\alpha)$ has more than one connected component than the correspondence between the thick part $K^{\text{thick}}(\varepsilon, \mathcal{C})$ of a connected component K of $\mathcal{H}_1(\alpha)$ and the principal boundary $\mathcal{H}(\alpha')$ is slightly more complicated. For example, to construct a surface T in $K^{\text{thick}}(\varepsilon, \mathcal{C})$, where K is a hyperelliptic component, we must start with a surface T' in the principal boundary such that all connected components T'_i of T' are hyperelliptic surfaces. So only the hyperelliptic components of some strata are in the principal boundary of K. It might even happen that only part of a connected component of a stratum is in the principal boundary. We still denote the principal boundary of $K^{\text{thick}}(\varepsilon, \mathcal{C})$ by $\mathcal{H}(\alpha')$, although $\mathcal{H}(\alpha')$ might be the union of (parts of) connected components of some strata.

We remark in passing that there are also some parameters used to describe the surgeries but this does not affect our computations.

If $q > 1$, then to parametrize the q tori we will always replace h_q by $h = h_1 + \cdots + h_q$. So the heights $(h_1, \ldots, h_{q-1})$ are in the cone $\Delta^{q-1}(h)$ given by the conditions $h_i > 0$, for $1 \leq i \leq q-1$, and $h_1 + \cdots + h_{q-1} < h$.

For a given $\varepsilon > 0$ we define

$$\begin{aligned}\mathcal{H}^\varepsilon(0^q) = \{&(\gamma, h, h_1, \ldots, h_{q-1}, t_1, \ldots, t_q) \mid \\ &\gamma \in B(\varepsilon), (h_1, \ldots, h_{q-1}) \in \Delta^{q-1}(h), (t_1, \ldots, t_q) \in [0, |\gamma|]^q\}.\end{aligned}$$

In what follows, $|\gamma|$ will always be small, but if no specific restriction on $|\gamma|$ is needed we will write $\mathcal{H}(0^q)$. We write $\mathrm{d}\nu(T)$ for the measure

$$\mathrm{d}\nu(T) = \mathrm{d}\gamma \, \mathrm{d}h \prod_{i=1}^{q-1} \mathrm{d}h_i \prod_{i=1}^{q} \mathrm{d}t_i.$$

We refer to the elements of $\mathcal{H}^\varepsilon(0^q)$ as tori as we obtain a torus with q marked points by joining the q cylinders. We write $\mathcal{H}_1^\varepsilon(0^q)$ for the subset of $\mathcal{H}^\varepsilon(0^q)$ of area 1 tori, meaning that they satisfy the condition $h|\gamma| = 1$.

We remark in passing that $\mathcal{H}^\varepsilon(0^q)$ can be interpreted as the ε -neighborhood of the "cusp" of the moduli space of flat tori with q marked points except that the marked points are already named by the way we parametrize them. This is in contrast to the cusp of the usual moduli space $\mathcal{H}^\varepsilon(\underbrace{0, \ldots, 0}_{q \text{ times}})$ where the marked points can be arbitrarily named.

We denote by $\mathrm{d}\nu(S)$ (resp. $\mathrm{d}\nu(S')$) the measure on $\mathcal{H}(\alpha)$ (resp. $\mathcal{H}(\alpha')$). It is shown in [EMZ] that

$$\mathrm{d}\nu(S) = \mathrm{d}\nu(S') \cdot \mathrm{d}\nu(T).$$

Let S be a translation surface in $\mathcal{H}_1(\alpha)$. For r a positive real number we denote by $rS \in \mathcal{H}(\alpha)$ the surface we get by multiplying the flat metric on S by r. Equivalently, if we represent S by an Abelian differential ω with respect to some complex structure, then rS corresponds to the Abelian differential $r\omega$ with respect to the same complex structure. Note that we have in particular $\text{area}(rS) = r^2 \, \text{area}(S)$.

If X is a subset of $\mathcal{H}_1(\alpha)$ then the cone $C(X)$ is defined to be

$$C(X) = \{rS \mid 0 < r < 1, S \in X\} \subset \mathcal{H}(\alpha).$$

If we denote by $\mathrm{d}\,\mathrm{vol}(S)$ the measure on $\mathcal{H}_1(\alpha)$ induced by the measure $\mathrm{d}\nu(S)$ then we have

$$\mathrm{d}\nu(S) = r^{\dim_{\mathbb{R}} \mathcal{H}(\alpha) - 1} \mathrm{d}r \; \mathrm{d}\,\mathrm{vol}(S).$$

So in particular one has

$$\mathrm{vol}(\mathcal{H}_1(\alpha)) = \dim_{\mathbb{R}} \mathcal{H}(\alpha) \, \mathrm{vol}(\mathcal{H}(\alpha)).$$

One has analogous statements for $\mathcal{H}_1(\alpha')$ and $\mathcal{H}_1(0^q)$ with its induced measures denoted by $\mathrm{d}\,\mathrm{vol}(S')$ and $\mathrm{d}\,\mathrm{vol}(T)$.

Note that there are $q+1$ complex parameters for $\mathcal{H}^{\varepsilon}(0^q)$ and that one has $\dim_{\mathbb{R}} \mathcal{H}(\alpha) = \dim_{\mathbb{R}} \mathcal{H}(\alpha') + \dim_{\mathbb{R}} \mathcal{H}(0^q)$. So by writing $n = \dim_{\mathbb{C}} \mathcal{H}(\alpha')$ we have

$$\begin{aligned} \dim_{\mathbb{R}} \mathcal{H}(\alpha') &= 2n \\ \dim_{\mathbb{R}} \mathcal{H}(0^q) &= 2(q+1) \\ \dim_{\mathbb{R}} \mathcal{H}(\alpha) &= 2(n+q+1) \end{aligned}$$

We recall for completeness that if $\alpha = (d_1, \ldots, d_m)$ with $d_i \geq 1$ for $i = 1, \ldots, m$, then

$$\dim_{\mathbb{R}} \mathcal{H}(\alpha) = 2(2g+m-1).$$

2.2 Siegel–Veech constants.

2.2.1 General method

We will describe in this section the method used to compute the Siegel–Veech constants introduced in section 2.1.2. Consider a stratum $\mathcal{H}(\alpha)$ of Abelian differentials where $\alpha = (d_1, \ldots, d_m)$ and $d_i \geq 1$ for $i = 1, \ldots, m$. Suppose that K is a connected component of $\mathcal{H}_1(\alpha)$ and that $\mathcal{C}$ is an admissible topological type of configurations for K containing exactly $q \geq 1$ cylinders. We denote the corresponding principal boundary stratum by $\mathcal{H}(\alpha')$. Recall that $K^{\text{thick}}(\varepsilon, \mathcal{C})$ is the set of translation surfaces in K that contain exactly one configuration of type $\mathcal{C}$ of length smaller than ε but do not contain another closed saddle connection of length smaller than ε. To simplify notation we will write from now on K^{ε} for $K^{\text{thick}}(\varepsilon, \mathcal{C})$.

Suppose that $N(S, \mathcal{C}, \varepsilon)$ is one of the above mentioned counting functions that counts configurations on S of length smaller than ε and of topological type $\mathbb{C}$. We want to evaluate $\int_{K^{\varepsilon}} N(S, \mathcal{C}, \varepsilon) \mathrm{d}\nu(S)$.

The decomposition of a surface $S \in \mathcal{H}(\alpha)$ as q tori with marked points on the boundary and a surface S' in $\mathcal{H}(\alpha')$ on which one performs surgeries (M choices) gives the following parametrisation of the cone $C(K^{\varepsilon})$: before performing the surgeries, a surface S in $C(K^{\varepsilon})$ is made up of sS_1' and tT_1 for some scalars s, t where $S_1' \in \mathcal{H}_1(\alpha')$ and $T_1 \in \mathcal{H}_1(0^q)$. The conditions are the following :
(i) the total area $s^2 + t^2$ of S satisfies $s^2 + t^2 \leq 1$;
The area 1 surface $\frac{1}{\sqrt{s^2+t^2}} S$ is in K^{ε} so the waist curve of $\frac{t}{\sqrt{s^2+t^2}} T_1$ is smaller than ε which means that the waist curve of T_1 is smaller than $\varepsilon \frac{\sqrt{s^2+t^2}}{t}$. So the second condition is:
(ii) T_1 has to lie in $\mathcal{H}_1^{\varepsilon'}(0^q)$, where $\varepsilon' = \varepsilon \frac{\sqrt{s^2+t^2}}{t}$.

We have

$$\int_{C(K^{\varepsilon})} N(S, \mathcal{C}, \varepsilon) \mathrm{d}\nu(S) =$$

$$= M \operatorname{vol}(\mathcal{H}_1(\alpha')) \int_0^1 s^{2n-1} \mathrm{d}s \int_0^{\sqrt{1-s^2}} t^{2q+1} \mathrm{d}t \int_{\mathcal{H}_1^{\varepsilon'}(0^q)} N(S, \mathcal{C}, \varepsilon) \mathrm{d} \operatorname{vol}(T) + o(\varepsilon^2), \quad (2.2)$$

where $\varepsilon' = \varepsilon'(s,t) = \frac{\varepsilon \cdot \sqrt{s^2+t^2}}{t}$. Note that if we construct $S \in C(K^{\varepsilon})$ as described above then S contains exactly one configuration of type $\mathcal{C}$ and of width less than ε, so $N(S, \mathcal{C}, \varepsilon)$ can be replaced by the weight with which we count configurations.

This integral is a simplification of the integral from [EMZ], page 133. In [EMZ] the statement is about the volume of $C(K^{\varepsilon})$ (in our notaton) as the authors only count configurations with weight one, so $N(S, \mathcal{C}, \varepsilon) = 1$ on K^{ε}.

2.2.2 Mean area of a cylinder

Fix an admissible configuration $\mathcal{C}$ in a connected component K of a stratum $\mathcal{H}_1(\alpha)$ that comes with $q \geq 1$ cylinders and let p be a real number $p \geq 0$.

Proof of Theorem 2. We choose and fix one of the named closed saddle connection of $\mathcal{C}$ that bounds a cylinder and call this cylinder the first one. We denote by $N_{\text{area}_1^p}(S, \mathcal{C}, \varepsilon)$ the number of configurations on S of type $\mathcal{C}$ of length at most ε, counted with weight the p-th power of the area of the first cylinder. We denote the corresponding Siegel-Veech constant by $c_{\text{area}_1^p}(K, \mathcal{C})$.

As described above, we decompose a surface S in $C(K^\varepsilon)$ as sS_1' and tT_1. We use the usual parameters $\gamma, h, h_1, \ldots, h_{q-1}, t_1, \ldots, t_q$ to parametrize $T_1 \in \mathcal{H}_1^\varepsilon(0^q)$. We assume the notation chosen so that the first cylinder of S corresponds to the first cylinder of T_1. So the area of the first cylinder of S is $t^2 h_1 w$, where $w = |\gamma|$. We use the following weight:

$$\left(\frac{t^2 h_1 w}{s^2 + t^2}\right)^p.$$

This weight is invariant under scaling of S and if the area $s^2 + t^2$ of S equals 1 then it reduces to $(t^2 h_1 w)^p$ which is the p-th power of the area of the first cylinder of S. A surface $S \in C(K^\varepsilon)$ contains exactly one configuration of type $\mathcal{C}$ of width at most ε, so we need to evaluate the integral of equation (2.2) for

$$N(S, \mathcal{C}, \varepsilon) = N_{\text{area}_1^p}(S, \mathcal{C}, \varepsilon) = \left(\frac{t^2}{s^2 + t^2}\right)^p (h_1 w)^p.$$

The first step to do this is to show that

Lemma 3.

$$\int_{\mathcal{H}_1^\varepsilon(0^q)} (h_1 w)^p \mathrm{d}\,\mathrm{vol}(T) = \frac{2\pi\varepsilon^2}{(p+1)\cdot(p+2)\cdots(p+q-1)},$$

where for $q = 1$ the denominator is by convention equal to 1.

Proof. For $q = 1$ we have $h_1 w = hw = 1$, so the computation is a simplified version of the general case. Assume that $q > 1$. We will first integrate over the domain $C(\mathcal{H}_1^\varepsilon(0^q))$, still using the parameters $\gamma, h, h_1, \ldots, h_{q-1}, t_1, \ldots, t_q$ for $T \in C(\mathcal{H}_1^\varepsilon(0^q))$. To have a weight that is invariant under scaling of T and that becomes $(h_1 w)^p$ for an area one torus T_1, we use the weight

$$\left(\frac{h_1 w}{hw}\right)^p = \left(\frac{h_1}{h}\right)^p.$$

The domain of integration for $T \in C(\mathcal{H}_1^\varepsilon(0^q))$ is as follows: The area wh of T satisfies $wh \leq 1$. We have $\frac{1}{\sqrt{wh}} T \in \mathcal{H}_1^\varepsilon(0^q)$, so the length $\frac{w}{\sqrt{wh}}$ of the waist curve of $\frac{1}{\sqrt{wh}} T$ is smaller than ε, which gives $h \geq \frac{w}{\varepsilon^2}$. (See figure 2.4.)

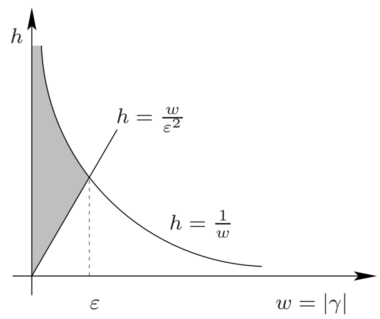


Figure 2.4: Domain of integration for $C(\mathcal{H}_1^\varepsilon(0))$

$$\int\limits_{C(\mathcal{H}_1^\varepsilon(0^q))} \left(\frac{h_1}{h}\right)^p \mathrm{d}\,\mathrm{vol}(T) = \int\limits_0^{2\pi} \mathrm{d}\theta \int\limits_0^{\varepsilon} w\,\mathrm{d}w \int\limits_{w/\varepsilon^2}^{1/w} \mathrm{d}h \int_0^h \left(\frac{h_1}{h}\right)^p \mathrm{d}h_1 \cdot$$

$$\cdot \int\limits_{\Delta^{q-2}(h-h_1)} \mathrm{d}h_2 \ldots \mathrm{d}h_{q-1} \int\limits_{[0,w]^q} \mathrm{d}t_1 \ldots \mathrm{d}t_q. \quad (2.3)$$

The volume of the cone $\Delta^{q-2}(h-h_1)$ is $\dfrac{(h-h_1)^{q-2}}{(q-2)!}$ and the volume of the cube $[0,w]^q$ is w^q. Using the change of variables $u = h_1/h$, the right hand side of equation (2.3) then evaluates to

$$\frac{\pi\varepsilon^2}{(q-2)!}\frac{\mathrm{B}(p+1,q-1)}{q+1} = \frac{\pi\varepsilon^2}{q+1}\frac{1}{(p+1)\cdots(p+q-1)},$$

where $\mathrm{B}(\cdot,\cdot)$ is the Beta function defined by (2.10). We get the right hand side using relations (2.11) and (2.9).

The weight $f(S) = (h_1/h)^p$ satisfies $f(rS) = f(S)$ and, if $S_1 \in \mathcal{H}_1^\varepsilon(0^q)$, $f(S_1) = (h_1 w)^p$, so

$$\int\limits_{C(\mathcal{H}_1^\varepsilon(0^q))} \left(\frac{h_1}{h}\right)^p \mathrm{d}\,\mathrm{vol}(T) =$$

$$= \int\limits_0^1 r^{2(q+1)-1}\mathrm{d}r \int\limits_{\mathcal{H}_1^\varepsilon(0^q)} f(rS_1)\mathrm{d}\,\mathrm{vol}(S_1) = \frac{1}{2(q+1)} \int\limits_{\mathcal{H}_1^\varepsilon(0^q)} (h_1 w)^p \mathrm{d}\,\mathrm{vol}(S_1),$$

which completes the proof of the lemma. □

To continue the proof of Theorem 2, we evaluate equation (2.2). We use lemma 3 with $\varepsilon' = \dfrac{\varepsilon\cdot\sqrt{s^2+t^2}}{t}$ and integrate the function $\left(\dfrac{t^2}{s^2+t^2}\right)^p$ that is the remaining part of $N_{\mathrm{area}_1^p}(S,\mathcal{C},\varepsilon)$. So equation (2.2) becomes

$$\int_{C(K^\varepsilon)} N_{\mathrm{area}_1^p}(S,\mathcal{C},\varepsilon)\mathrm{d}\nu(S) = \frac{M\cdot\mathrm{Vol}(\mathcal{H}_1(\alpha'))\cdot 2\pi\varepsilon^2}{(p+1)\cdot(p+2)\cdots(p+q-1)} J_p + o(\varepsilon^2),$$

where

$$J_p = \int_0^1 s^{2n-1}\mathrm{d}s \int_0^{\sqrt{1-s^2}} t^{2q+1}\cdot\left(\frac{t^2}{s^2+t^2}\right)^{p-1}\mathrm{d}t$$

Using polar coordinates $s = r\cos\theta$, $t = r\sin\theta$ we get

$$J_p(\mathcal{C}) = \frac{1}{2(n+q+1)}\int_0^{\frac{\pi}{2}} (\cos\theta)^{2n-1}(\sin\theta)^{2p+2q-1}\mathrm{d}\theta.$$

Using $u = \cos^2\theta$ we get

$$J_p(\mathcal{C}) = \frac{\mathrm{B}(n,q+p)}{4(n+q+1)} = \frac{1}{4(n+q+1)}\frac{(n-1)!}{(p+q)\cdots(p+q+n-1)}. \tag{2.4}$$

Using the fact that for $S \in C(K^\varepsilon)$, $N_{\mathrm{area}^p}(S,\mathcal{C},\varepsilon) = \left(\frac{t^2}{s^2+t^2}\right)^p$ is invariant under scaling of S and reduces to the initial definition of $N_{\mathrm{area}^p}(S_1,\mathcal{C},\varepsilon)$ if $S_1 \in K^\varepsilon$, we can show as in the proof of lemma 3 that

$$\int\limits_{K^\varepsilon} N_{\mathrm{area}_1^p}(S_1,\mathcal{C},\varepsilon)\mathrm{d}\,\mathrm{vol}(S_1) = 2(n+q+1)\int\limits_{C(K^\varepsilon)} N_{\mathrm{area}_1^p}(S,\mathcal{C},\varepsilon)\mathrm{d}\nu(S),$$

where $2(n+q+1) = \dim_{\mathbb{R}}\mathcal{H}(\alpha)$. And so

$$\int\limits_{K^\varepsilon} N_{\mathrm{area}_1^p}(S,\mathcal{C},\varepsilon)\mathrm{d}\,\mathrm{vol}(S) =$$

$$= M\cdot\pi\varepsilon^2\cdot\mathrm{Vol}(\mathcal{H}_1(\alpha'))\cdot\frac{(n-1)!}{(p+1)\cdots(p+q+n-1)} + o(\varepsilon^2).$$

Using equation (2.1) we conclude that

$$c_{\mathrm{area}_1^p}(K,\mathcal{C}) = M\cdot\frac{\mathrm{Vol}(\mathcal{H}_1(\alpha'))}{\mathrm{Vol}(K)}\cdot\frac{(n-1)!}{(p+1)\cdots(p+q+n-1)}.$$

Note that if we define $N_{\mathrm{area}_i^p}(S,\mathcal{C},\varepsilon)$ in the same way as $N_{\mathrm{area}_1^p}(S,\mathcal{C},\varepsilon)$, except that we use the area of the i-th cylinder, for $i=1,\cdots,q$, then we have $N_{\mathrm{area}^p}(S,\mathcal{C},\varepsilon)=\sum_i N_{\mathrm{area}_i^p}(S,\mathcal{C},\varepsilon)$. It follows that $c_{\mathrm{area}^p}(K,\mathcal{C}) = q c_{\mathrm{area}_1^p}(K,\mathcal{C})$, which completes the proof of Theorem 2. □

Remark. (a) We note that the volume of the stratum $\mathcal{H}_1(\alpha')$ of disconnected surfaces was computed in [EMZ], equation (12); writing $\mathcal{H}(\alpha') = \Pi_{i=1}^p\mathcal{H}(\alpha'_i)$ and $n_i = \dim_{\mathbb{C}}\mathcal{H}(\alpha'_i)$, we have

$$\mathrm{Vol}(\mathcal{H}_1(\alpha')) = \frac{1}{2^{p-1}}\cdot\frac{\prod_{i=1}^p(n_i-1)!}{(n-1)!}\cdot\prod_{i=1}^p\mathrm{Vol}(\mathcal{H}_1(\alpha'_i)).$$

(b) As an example we consider the moduli space of tori. To have a configuration we need to mark one regular point. The only possible topological type $\mathcal{C}$ of configuration is a closed saddle connection based at this regular point. We get, using Lemma 3 for $p=0$ and $q=1$,

$$c_{\mathrm{conf}}(\mathcal{H}_1(0),\mathcal{C}) = \lim_{\varepsilon\to 0}\frac{1}{\pi\varepsilon^2}\cdot\frac{\mathrm{Vol}(\mathcal{H}_1^\varepsilon(0))}{\mathrm{Vol}(\mathcal{H}_1(0))} = \frac{1}{\pi\varepsilon^2}\cdot\frac{2\pi\varepsilon^2}{\pi^2/3} = \frac{6}{\pi^2} = \frac{1}{\zeta(2)}$$

We got the well-known factor for the proportion of coprime lattice points in $\mathbb{Z}\oplus\mathbb{Z}$.

2.2.3 Mean area of the periodic region

Fix an admissible topological type of configuration $\mathcal{C}$ for a connected component K of a stratum $\mathcal{H}_1(\alpha)$ that comes with $q\geq 1$ cylinders. Recall that $N_{\mathrm{area}^p,\mathrm{conf}}(S,\mathcal{C},L)$ denotes the p-th power of the total area of the periodic region (union of the cylinders) on S coming from a configuration of topological type $\mathcal{C}$ whose length is at most L.

Proof of theorem 3. The argument is a special case of the argument in section 2.2.2. By decomposing as before a surface S in $C(K^\varepsilon)$ as sS'_1 and tT_1 we use the weight

$$N_{\mathrm{area}^p,\mathrm{conf}}(S,\mathcal{C},\varepsilon) = \left(\frac{t^2}{s^2+t^2}\right)^p.$$

So by taking $p=0$ in lemma 3,

$$\int_{\mathcal{H}_1^\varepsilon(0^q)} \mathrm{d}\,\mathrm{vol}(T) = \frac{2\pi\varepsilon^2}{(q-1)!}. \tag{2.5}$$

We then have

$$\int_{C(K^\varepsilon)} N_{\mathrm{area}^p,\mathrm{conf}}(S,\mathcal{C},\varepsilon)\mathrm{d}\nu(S) = M\cdot\mathrm{Vol}(\mathcal{H}_1(\alpha'))\cdot\frac{2\pi\varepsilon^2}{(q-1)!}J_p + o(\varepsilon^2),$$

where J_p satisfies relation (2.4). We conclude as in section 2.2.2 that

$$\int_{K^\varepsilon} N_{\mathrm{area}^p,\mathrm{conf}}(S,\mathcal{C},\varepsilon)\mathrm{d}\,\mathrm{vol}(S) = \frac{M\cdot\pi\varepsilon^2\cdot\mathrm{Vol}(\mathcal{H}_1(\alpha'))\cdot(n-1)!}{(q-1)!\cdot(p+q)\cdots(p+q+n-1)} + o(\varepsilon^2).$$

It suffices to apply relation (2.1). □

2.2.4 Configurations with periodic regions of large area.

Fix an admissible configuration $\mathcal{C}$ in a connected component K of a stratum $\mathcal{H}_1(\alpha)$ that comes with $q \geq 1$ cylinders and let $p \in [0,1)$ be a real parameter. Recall that $N_{\mathrm{conf},A\geq p}(S,\mathcal{C},\varepsilon)$ denotes the number of configurations on S of type $\mathcal{C}$ of length smaller than ε and such that the total area of the q cylinders is at least p (of the area one surface S).

Proof of Theorem 5. We use a modification of the argument from section 2.2.2. Here we consider the subset of K^ε consisting of surfaces S whose area of the periodic part is at least p (of the area 1 surface S). Construct S in the cone of this set using sS_1' and tT_1 for some scalars s,t and $S_1' \in \mathcal{H}_1(\alpha')$ and $T_1 \in \mathcal{H}_1^{\varepsilon'}(0^q)$. We need to evaluate the integral in equation (2.2) with an additional condition : when scaling S by $\frac{1}{s^2+t^2}$ we get a surface of area one that satisfies

$$\operatorname{area}\left(\frac{t}{\sqrt{s^2+t^2}}T_1\right) = \frac{t^2}{s^2+t^2} > p \qquad \Longleftrightarrow \qquad t \geq \sqrt{\frac{p}{1-p}}\, s.$$

We count a configuration that satisfies this additional constraint with weight 1, so equation (2.2) becomes

$$\int\limits_{C(K^\varepsilon)} N_{\mathrm{conf},A>p}(S,\mathcal{C},\varepsilon)\mathrm{d}\nu(S)$$

$$= M \operatorname{vol}(\mathcal{H}_1(\alpha')) \int\limits_0^1 s^{2n-1}\mathrm{d}s \int\limits_{\sqrt{\frac{p}{1-p}}s}^{\sqrt{1-s^2}} t^{2q+1}\mathrm{d}t \int\limits_{\mathcal{H}_1^{\varepsilon'}(0^q)} \mathrm{d}\operatorname{vol}(T) + o(\varepsilon^2),$$

where $\varepsilon' = \dfrac{\varepsilon\cdot\sqrt{s^2+t^2}}{t}$.

Using relation (2.5), the right hand side becomes

$$\frac{M\cdot 2\pi\varepsilon^2}{(q-1)!}\cdot \operatorname{Vol}(\mathcal{H}_1(\alpha'))\cdot I_p + o(\varepsilon^2)$$

where

$$I_p = \int_0^{\sqrt{1-p}} s^{2n-1} \int_{\sqrt{\frac{p}{1-p}}s}^{\sqrt{1-s^2}} t^{2q+1}\frac{s^2+t^2}{t^2}\mathrm{d}t\mathrm{d}s.$$

The domain of integration for s and t is described in Figure 2.5.

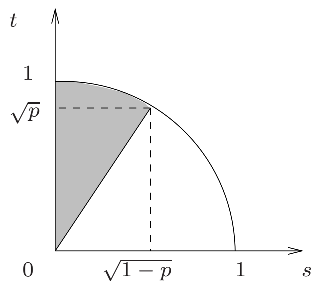


Figure 2.5: Domain of integration.

Using polar coordinates $s = r\cos\theta$, $t = r\sin\theta$ and setting $\alpha = \arccos\sqrt{1-p}$ we get

$$I_p = \frac{1}{2(n+q+1)}\int_\alpha^{\frac{\pi}{2}} (\cos\theta)^{2n-1}(\sin\theta)^{2q-1}\mathrm{d}\theta.$$

Using the change of variables $u = \cos^2\theta$ we get

$$I_p = \frac{1}{4(n+q+1)} \int_0^{1-p} u^{n-1}(1-u)^{q-1}\mathrm{d}u = \frac{B(1-p;n,q)}{4(n+q+1)},$$

where $B(\cdot;\cdot,\cdot)$ is the incomplete Beta function as defined in equation (2.12).

Note that we have $c_{\mathrm{conf}}(K,\mathcal{C}) = c_{\mathrm{conf},A\geq 0}(K,\mathcal{C})$, so up to the same constant, the integral used to compute $c_{\mathrm{conf},A\geq 0}(K,\mathcal{C})$, resp $c_{\mathrm{conf}}(K,\mathcal{C})$ is given by I_p, resp I_0 (or equivalently J_0, see equation (2.4)), so

$$\frac{c_{\mathrm{conf},A\geq p}(K,\mathcal{C})}{c_{\mathrm{conf}}(K,\mathcal{C})} = \frac{I_p}{I_0} = \mathrm{I}(1-p;n,q).$$

We proved Theorem 5 of section 2.1.2, where, to expand $\mathrm{I}(1-p;n,q)$, we use Lemma 8. $\square$

2.2.5 Configurations with a cylinder of large area

Fix an admissible configuration $\mathcal{C}$ in a connected component K of a stratum $\mathcal{H}_1(\alpha)$ that comes with $q \geq 1$ cylinders and let p be a parameter that satisfies $0 \leq p < 1$.

We choose and fix one of the named closed saddle connection of $\mathcal{C}$ that bounds a cylinder and call this cylinder the first one. Recall that $N_{\mathrm{conf},A_1\geq p}(S,\mathcal{C},L)$ is the number of configurations of type $\mathcal{C}$ of length at most L and such that the area of the first cylinder is at least p (of the area one surface S).

Proof of theorem 4. Suppose that we have $q \geq 1$ cylinders. The argument is as in section 2.2.4 with the following modification: We replace "area" by "area of the first cylinder" in the condition $\mathrm{area}(\frac{t}{\sqrt{s^2+t^2}}T_1) = \frac{t^2}{s^2+t^2} > p$. Denote by $Cusp_a(\varepsilon')$ the subset of $\mathcal{H}_1^{\varepsilon'}(0^q)$ of tori T_1 such that the area of the first cylinder of T_1 is at least $a = p\frac{s^2+t^2}{t^2}$. Using the usual parameters (γ, h, h_i, t_j), a torus T is in the cone of $Cusp_a(\varepsilon')$ if the area one torus $\frac{1}{\sqrt{wh}}T$ is in $Cusp_a(\varepsilon')$, so the parameters for T have the additional constraint $\frac{wh_1}{wh} > a$.

To compute the volume of $Cusp_a(\varepsilon')$ we proceed as in the proof of lemma 3 for $p = 0$. The only difference is that we replace the condition $0 \leq h_1 \leq h$ by $ah \leq h_1 \leq h$. So $(h_1, \dots, h_{q-1})$ is in a cone whose volume is $\frac{((1-a)h)^{q-1}}{(q-1)!}$. So the volume is as in equation (2.5) except that we have an extra factor $(1-a)^{q-1}$. So we need to integrate

$$I'_p = \int_0^{\sqrt{1-p}} s^{2n-1} \int_{\sqrt{\frac{p}{1-p}}s}^{\sqrt{1-s^2}} t^{2q+1}(1-a)^{q-1}\frac{s^2+t^2}{t^2}\,\mathrm{d}t\mathrm{d}s. \tag{2.6}$$

Using polar coordinates $s = r\cos\theta$, $t = r\sin\theta$ followed by the change of variables $w = \frac{\cos^2\theta}{1-p}$ we get

$$I'_p = \frac{(1-p)^{n+q-1}}{4(n+q+1)} B(n,q). \tag{2.7}$$

Note that $c_{\mathrm{conf}}(K,\mathcal{C}) = c_{\mathrm{conf},A_1\geq 0}(K,\mathcal{C})$, so $\frac{c_{\mathrm{conf},A_1\geq p}(K,\mathcal{C})}{c_{\mathrm{conf}}(K,\mathcal{C})} = \frac{I'_p}{I'_0}$ as claimed. (Recall that $\dim_{\mathbb{C}}\mathcal{H}(\alpha) = n+q+1$.) $\square$

Remark. For $q = 1$ we have $c_{\mathrm{conf},A_1\geq p}(K,\mathcal{C}) = c_{\mathrm{conf},A\geq p}(K,\mathcal{C})$ (see Theorem 5).

2.2.6 Correlation between the area of two cylinders.

Let $\mathcal{C}$ be an admissible configuration for a connected component K that comes with at least two cylinders. Choose (and fix) two cylinders and let $p, p_1 \in [0,1)$. Recall that we denote by $N_{A_2\geq p,A_1\geq p_1}(S,\mathcal{C},L)$ the number of configurations of length at least L such that the area A_1 of the first cylinder is at least p_1 and such that the area A_2 of the second cylinder is at least $p(1-A_1)$. We denote by $c_{A_2\geq p,A_1\geq p_1}(K,\mathcal{C})$ the corresponding Siegel–Veech constant. To simplify notation we will write $c_{A_1\geq p_1}(K,\mathcal{C})$ instead of $c_{\mathrm{conf},A_1\geq p_1}(K,\mathcal{C})$.

Proof of theorem 6. We proceed as in the proof of Theorem 4 (see section 2.2.5). Using the same notation, we have the condition that the area $\tilde{A}_1$ of the first cylinder of $\frac{t}{\sqrt{s^2+t^2}}T_1$ is at least p_1 and the area $\tilde{A}_2$ of the second cylinder of $\frac{t}{\sqrt{s^2+t^2}}T_1$ is at least $p(1-\tilde{A}_1)$. This means that the area A_1 of the first cylinder of T_1 satisfies $A_1 \geq a_1 = p_1\frac{s^2+t^2}{t^2}$ and the area A_2 of the second cylinder is at least $p(1-\tilde{A}_1)\frac{s^2+t^2}{t^2}$ which gives $A_2 \geq p\left(\frac{s^2+t^2}{t^2} - A_1\right) = a - pA_1$, where $a = p\frac{s^2+t^2}{t^2}$.

Denote by $\mathrm{Cusp}_{a_1,a}(\varepsilon')$ the subset of tori in $\mathcal{H}_1^{\varepsilon'}(0^q)$ that satisfies these conditions. Using the usual parameters (γ, h, h_i, t_j), a torus T is in the cone $C(\mathrm{Cusp}_{a_1,a}(\varepsilon'))$ if and only if $T_1 = \frac{1}{\sqrt{wh}}T$ is in $\mathrm{Cusp}_{a_1,a}(\varepsilon')$. So the areas A_1 and A_2 of the first and second cylinders of T_1 must satisfy $A_1 = \frac{wh_1}{wh} \geq a_1$ and $A_2 = \frac{wh_2}{wh} \geq a - p\frac{wh_1}{wh}$. So we get

$$a_1 h \leq h_1 \leq h, \quad (*) \qquad\qquad ah - ph_1 \leq h_2 \leq h - h_1. \quad (**)$$

Equation $(**)$ has a solution if and only if $ah - ph_1 \leq h - h_1$ which can be written as $h_1 \leq \frac{1-a}{1-p}h$ so we need to modify $(*)$:

$$a_1 h \leq h_1 \leq \frac{1-a}{1-p}h. \quad (*')$$

Equation $(*')$ has a solution if and only if $a_1 \leq \frac{1-a}{1-p}$. This translates into $p_2\frac{s^2+t^2}{t^2} \leq 1$, where $p_2 = p + p_1(1-p)$, which in turn becomes

$$t \geq \sqrt{\frac{p_2}{1-p_2}}s.$$

We also have $0 \leq h_1 \leq h$ and $0 \leq h_2 \leq h - h_1$. But for all possible t, s we have $a_1 \geq 0$, $ah - ph_1 \geq 0$, and $\frac{1-a}{1-p} \leq 1$, so h_1 can take all values between $a_1 h$ and $\frac{1-a}{1-p}h$ and h_2 can take all values between $ah - ph_1$ and $h - h_1$.

The computation of the volume of $\mathrm{Cusp}_{a_1,a}(\varepsilon')$ is as in the proof of Lemma 3 for $p = 0$, except that we replace

$$\int_{\Delta^{q-1}(h)} \mathrm{d}h_1 \ldots \mathrm{d}h_{q-1} = \frac{h^{q-1}}{(q-1)!}$$

by

$$\int_{a_1 h}^{\frac{1-a}{1-p}h} \mathrm{d}h_1 \int_{ah-ph_1}^{h-h_1} \mathrm{d}h_2 \int_{\Delta^{q-3}(h-h_1-h_2)} \mathrm{d}h_3 \ldots \mathrm{d}h_{q-1}$$
$$= \frac{[(1-a)-(1-p)a_1]^{q-1}}{1-p}\frac{h^{q-1}}{(q-1)!} = \frac{(1-a_2)^{q-1}}{1-p}\frac{h^{q-1}}{(q-1)!},$$

where $a_2 = p_2\frac{s^2+t^2}{t^2}$.

So the volume is as in equation (2.5) except that we have an extra factor $\frac{(1-a_2)^{q-1}}{1-p}$. So we need to integrate

$$\frac{1}{1-p}\int_0^{\sqrt{1-p_2}} s^{2n-1} \int_{\sqrt{\frac{p_2}{1-p_2}}s}^{\sqrt{1-s^2}} t^{2q+1}(1-a_2)^{q-1}\frac{s^2+t^2}{t^2}\,\mathrm{d}t\mathrm{d}s.$$

Note that up to the factor $1/(1-p)$ this is I'_{p_2} as defined in (2.6). The integral used to compute $c_{X_1 \geq p_1}(K, \mathcal{C}) = c_{\mathrm{conf}, X_1 \geq p_1}(K, \mathcal{C})$ is I'_{p_1}, so we conclude

$$\frac{c_{A_2 \geq p, A_1 \geq p_1}(K, \mathcal{C})}{c_{A_1 \geq p_1}(K, \mathcal{C})} = \frac{I'_{p_2}}{1-p} \cdot \frac{1}{I'_{p_1}}.$$

Using equation (2.7) and $1 - p_2 = (1-p)(1-p_1)$ we find

$$\frac{c_{A_2 \geq p, A_1 \geq p_1}(K, \mathcal{C})}{c_{A_1 \geq p_1}(K, \mathcal{C})} = \frac{(1-p_2)^{n+q-1}}{(1-p)(1-p_1)^{n+q-1}} = (1-p)^{n+q-2}.$$

$\square$

2.3 Extremal properties of configurations

2.3.1 Maximal total mean area of a configuration

Consider an admissible topological type $\mathcal{C}$ of configuration for some connected component K of some stratum $\mathcal{H}(\alpha)$, where $\alpha = (d_1, \ldots, d_m)$ satisfies $d_i \geq 1$, for $i = 1, \ldots, m$. We denote by $q(\mathcal{C})$ the number of cylinders that come with $\mathcal{C}$. In this section we prove Theorem 7, so we look for a topological type of configuration $\mathcal{C}$ (that is admissible for some connected component K) that maximizes

$$c_{\text{mean area conf}}(K, \mathcal{C}) = \frac{c_{\text{area}}(K, \mathcal{C})}{c_{\text{conf}}(K, \mathcal{C})} = \frac{q(\mathcal{C})}{2g + m - 2},$$

where for the last equality we used Corollary 1 and $c_{\text{cyl}}(K, \mathcal{C}) = q(\mathcal{C}) c_{\text{conf}}(K, \mathcal{C})$.

For a given stratum $\mathcal{H}(\alpha)$, we start by determining the maximal possible number of cylinders $q_{\max}(\alpha)$ that can come from an admissible topological type of configuration for $\mathcal{H}(\alpha)$:

$$q_{\max}(\alpha) = \max_{\mathcal{C} \text{ in } \mathcal{H}(\alpha)} q(\mathcal{C}).$$

It is shown in [EMZ] that topological types of configurations can be constructed by creating singular points of the following three types:

a. a cylinder, followed by $k \geq 1$ surfaces S_i of genus $g_i \geq 1$ with figure eight boundary, followed by a cylinder. See figure 2.6 for $k = 3$ and $g_i = 1$, $i = 1, 2, 3$. We say that the newborn singularity is of type I.

b. a cylinder, followed by $k \geq 0$ surfaces S_i of genus $g_i \geq 1$ with figure eight boundary, followed by a surface S_{k+1} of genus $g_{k+1} \geq 1$ with a pair of holes boundary. See Figure 2.7 for $k = 2$ and $g_i = 1$, $i = 0, 1, 2$. We say that the newborn singularity is of type II. For $k = 0$ we just have a cylinder followed by a surface with a pair of holes boundary. We might also reverse the order: a pair of holes torus followed by $k \geq 0$ figure eight tori followed by a cylinder.

c. a singularity of type III, which is obtained from a singularity of type II by replacing the cylinder by a surface with a pair of holes boundary. As for surfaces of type II there might be no surface with a figure eight boundary. See Figure 2.8 where all of the surfaces are of genus 1 and where we have two surfaces with figure eight boundary.

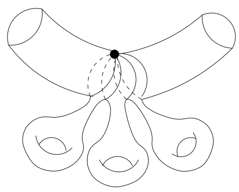


Figure 2.6: Block of surfaces creating a zero of type I

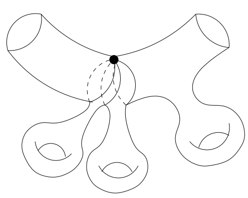


Figure 2.7: Block of surfaces creating a zero of type II

Counting angles, it is shown in [EMZ] that the order of the newborn zeros is as follows:

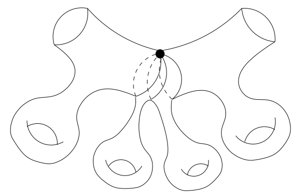


Figure 2.8: Block of surfaces creating a zero of type III

a. To create a zero of type I one uses $k \geq 1$ figure eight boundaries that were created at zeros of orders $a_1 \geq 0, \ldots, a_k \geq 0$ (a zero of order 0 being a regular point). The zero then has order $\sum_{i=1}^{k}(a_i+2)$. All orders bigger or equal to 2 are possible.

b. To create a zero of type II one uses a figure eight boundary that was created at a zero of order $b' \geq 0$. If there is no figure eight boundary involved then the newborn zero has order $b'+1$. If there are $k \geq 1$ figure eight boundaries involved that were created at zeros of orders $a_1 \geq 0, \ldots, a_k \geq 0$ then the newborn zero has order $(b'+1)+\sum_{i=1}^{k}(a_i+2)$. All orders bigger or equal to 1 are possible.

c. To create a zero of type III we use two pair of holes boundaries created at zeros of orders b_1' and b_2''. If there are no figure eight boundaries involved then the order of the newborn zero is $(b_1'+1)+(b_2''+1)$. If there are $k \geq 1$ figure eight boundaries involved that were created at zeros of orders $a_1 \geq 0, \ldots, a_k \geq 0$ then the newborn zero has order $(b'+1)+\sum_{i=1}^{k}(a_i+2)+(b_2''+1)$. All orders bigger or equal to 2 are possible.

For a given small γ one can create all of the boundaries of the surfaces involved in the above construction by an appropriate figure eight surgery or a pair of holes surgery (see Figures 2.2 and 2.3) such that the boundaries all have holonomy vector γ.

By arranging the blocks in a cyclic order and identify boundary components we create an admissible topological type of configuration of homologous saddle connections where each saddle connection is based at a newborn singularity of one of the three types. There is only the following obstruction: one needs to either use at least one surface with a pair of holes boundary or, if one only uses surfaces with figure eight boundaries, then one needs to use at least one cylinder.

Each cylinder is bounded by two saddle connections, so to have a one to one correspondence we think of the cylinder as being cut into two parts by the central waist curve. Each half cylinder then has a saddle connection γ on the boundary (that is not the waist curve) that joins a saddle P to itself. We say that P accounts for this half-cylinder. A zero of type I (see Figure 2.6) accounts for two half-cylinders (so for one cylinder), a zero of type II (see Figure 2.7) accounts for one half of a cylinder, and a zero of type III (see Figure 2.8) does not account for any half cylinder. Note that only singularities of type II can have order 1. It follows that if for $n \in \mathbb{N}$ we define

$$\chi(n) = \begin{cases} 1/2, & \text{if } n = 1 \\ 1, & \text{if } n > 1 \end{cases}$$

then we have $q_{\max}(d_1, \ldots, d_k) \leq \sum_{i=1}^{m}\chi(d_i)$. To maximize the number of cylinders we construct the zeros of orders greater than or equal to 2 by zeros of type I. The other zeros of order 1 must be created by zeros of type II (coming from a cylinder followed by a torus with a pair of holes boundary). This works well if there is an even number of zeros of order 1 in which case we constructed a configuration with $\sum_{i=1}^{n}\chi(d_i)$ cylinders. If there is an odd number of zeros of order 1 then the construction of the zeros of order 1 ends with a surface with a pair of holes boundary. In this case we are obliged to construct one of the zeros of order greater than 1 by a surface of type II, so we constructed a configuration with $\sum_{i=1}^{m}\chi(d_i) - 1/2$ cylinders. We showed

Proposition 4. *Consider a stratum $\mathcal{H}(\alpha)$, where $\alpha = (d_1, \ldots, d_m)$ satisfies $d_i \geq 1$, for $i = 1, \ldots, m$. We have*

$$q_{\max}(\alpha) = \left\lfloor \sum_{i=1}^{n}\chi(d_i) \right\rfloor, \tag{2.8}$$

where the square brackets denote the integer part of a number.

Remark. The proposition says that it is possible to find in each stratum $\mathcal{H}(\alpha)$ a topological type of configuration $\mathcal{C}$ such that $q(\mathcal{C}) = q_{\max}(\alpha)$ is as stated. It is not possible to find a topological type $\mathcal{C}$ in each *connected component* of $\mathcal{H}(\alpha)$ with $q_{\max}(\alpha)$ cylinders.

We next show that given any connected component K of a stratum $\mathcal{H}(\alpha)$ and any admissible topological type $\mathcal{C}$ of configuration for K,

$$c_{\text{mean area conf}}(K, \mathcal{C}) \leq \frac{1}{3}.$$

Proof. Let $g \geq 2$. Denoting by $\ell(\alpha)$ the length of α we have

$$c_{\text{mean area conf}}(K, \mathcal{C}) = \frac{q(\mathcal{C})}{2g - 2 + \ell(\alpha)} \leq \frac{q_{\max}(\alpha)}{2g - 2 + \ell(\alpha)}.$$

We determine

$$\max_{\alpha \in \Pi(2g-2)} \frac{q_{\max}(\alpha)}{2g - 2 + \ell(\alpha)},$$

where $\Pi(2g-2)$ denotes the set of permutations of $2g - 2$.

First let us prove that a partition α containing at least one entry 1 is not maximizing. Indeed, if there is at least one *pair* of entries 1 we can modify the initial partition α by replacing the elements $1, 1$ with a single entry 2. This does not change the value (2.8) of $q_{\max}(\alpha)$, but decreases the denominator in the ratio $q_{\max}/(2g - 2 + \ell(\alpha))$.

If there is a *single* entry 1 in the partition α, then $\sum_{i=1}^{n} \chi(d_i)$ is not an integer. We can modify α by deleting the entry 1 and increasing some other entry by 1. This operation does not change the value (2.8) of $q_{\max}(\alpha)$, but decreases the denominator in the ratio $q_{\max}/(2g - 2 + \ell(\alpha))$.

Thus we have proved that all of the entries of the partition maximizing the ratio $q_{\max}(\alpha)/2g - 2 + \ell(\alpha)$ are greater then or equal to 2. The formula (2.8) for such partitions simplifies to $q_{\max}(\alpha) = \ell(\alpha)$. Now note that for any $g \geq 2$ the function

$$f_g(x) = \frac{x}{2g - 2 + x} = 1 - \frac{2g - 2}{2g - 2 + x}$$

is strictly monotonously decreasing. Thus, among all partitions of $2g - 2$ with entries strictly greater than one we have to chose the one maximizing $\ell(\alpha)$. This is the partition $(2, \ldots, 2)$ where the order 2 appears $g - 1$ times. For this partition we get $q_{\max}(2, \ldots, 2) = g - 1 = \ell(\alpha)$ and so $c_{\text{mean area conf}}(K, \mathcal{C})$ is bounded from above by $1/3$ as claimed. $\square$

Proof of Theorem 7. It suffices to show that for each genus g there is an admissible topological type of configuration $\mathcal{C}$ for a connected component K of the stratum $\mathcal{H}(2, \ldots, 2)$ ($g - 1$ zeros of order 1) such that $c_{\text{mean area conf}}(K, \mathcal{C})$ attains the upper bound $1/3$.

Note that for the upper bound $1/3$ for $c_{\text{mean area conf}}(K, \mathcal{C})$ we used $q_{\max}$. As we explained in the proof of the relation (2.8), the only way to obtain $q_{\max}(2, \ldots, 2)$ is to only use zeros of type I (a cylinder followed by a torus with a figure eight boundary that was created at a regular point followed by a cylinder). Doing this we obtain a surface in $\mathcal{H}(2, \ldots, 2)$ with a configuration that comes with $g - 1$ cylinders.

By Lemma 14.2 in [EMZ] the surface in $\mathcal{H}(2, \ldots, 2)$ we constructed that comes with a maximizing configuration has odd parity of spin structure, so this surface is in $\mathcal{H}^{\text{odd}}(2, \ldots, 2)$. (We recall in the next section the classification of connected components.) Proposition 7 is proved. $\square$

2.3.2 Configurations with simple complementary regions

This section provides an answer to the following question of Alex Eskin and Alex Wright: is it possible to find in each connected component of a stratum an admissible topological type of configuration whose complementary regions are tori (with boundary) and cylinders. Motivations for this problem can be found in [Wr3].

The answer depends on the connected component, so we need to recall the classification of connected components for strata $\mathcal{H}(\alpha)$ of Abelian differentials from [KZ]. Some connected components are characterized by the fact that they only contain hyperelliptic surfaces. For a surface S in a stratum $\mathcal{H}(d_1, \ldots, d_n)$ where all d_i are even one has the notion of *parity of spin structure* that is either 0 or 1. We then have

Theorem (M. Kontsevich, A. Zorich [KZ]). *Let $\mathcal{H}(d_1, \ldots, d_m)$ be a stratum of Abelian differentials on a surface of genus $g \geq 4$. The strata $\mathcal{H}(2g-2)$ and $\mathcal{H}(g-1, g-1)$ are the only strata to have a hyperelliptic component $\mathcal{H}^{\text{hyp}}(2g-2)$, resp. $\mathcal{H}^{\text{hyp}}(g-1, g-1)$. Apart from these hyperelliptic components we have:*

a. *If at least one of the d_i is odd then there is only one non-hyperelliptic component.*

b. *If all of the α_i are even then there are two non-hyperelliptic components, $\mathcal{H}^{\text{even}}$ with even and $\mathcal{H}^{\text{odd}}$ with odd parity of spin structure.*

Remark. The classification in [KZ] also covers $g = 2, 3$ but we will not need this.

Proposition 5. *Let $\mathcal{H}^{comp}(\alpha)$ denote a connected component of a stratum of Abelian differentials on a surface of genus $g \geq 5$.*
Then:

a. *If $\mathcal{H}^{comp}(\alpha)$ is hyperelliptic (so $\mathcal{H}^{\text{hyp}}(2g-2)$ or, if $g-1$ is even, $\mathcal{H}^{\text{hyp}}(g-1, g-1)$) then it is not possible to find an admissible topological type of configuration whose complementary regions are only tori (with boundary) and cylinders.*

b. *If g is even and $\mathcal{H}^{comp}(\alpha) = \mathcal{H}^{even}(\alpha)$ then we can find a topological type of configuration whose complementary regions are tori, cylinders and one surface of genus two. But it is not possible to only have tori and cylinders.*

c. *In all remaining connected components one can explicitly construct an admissible topological type of configuration whose complementary regions are tori and cylinders.*

Proof. First note that a configuration containing only tori and cylinders, or tori, cylinders, and at most one genus two surface does not occur in hyperelliptic strata, since a hyperelliptic surface can contain at most two closed homologous saddle connections, which rules out $g \geq 5$. Lemma 14.5 in [EMZ] describes precisely configurations in hyperelliptic components of strata.

We suppose for what follows that $\mathcal{H}^{comp}(\alpha)$ is not a hyperelliptic component.

Recall the description of singularities of types I, II, and III from the previous section. If apart from cylinders, we only use surfaces of genus 1, then we must do figure eight surgeries and pair of holes surgeries at regular marked points. So we have:

a. A zero of type I has order $2k$, where $k \geq 1$ is the number of tori with figure eight boundaries (see Figure 2.6).

b. A zero of type II has order $2k+1$, where $k \geq 0$ is the number of tori with figure eight boundaries (see figure 2.7).

c. A zero of type III has order $2k+2$, where $k \geq 0$ is the number of tori with figure eight boundaries (see figure 2.8).

Assume that α contains at least one odd d_i, so $\alpha = (2a_1, \cdots, 2a_p, 2b_1+1, \cdots, 2b_r+1)$ with $p \geq 0$ and $r \geq 1$ ($p = 0$ corresponds to the case when all α_i are odd).

We construct blocs of surfaces that contain zeros of type II of orders $2b_1+1, \ldots, 2b_r+1$. Note that since $\sum_i d_i$ is even, the number r of odd d_i is even, so in our construction the first and last surface is a cylinder. If there is at least one zero of even order then we construct in addition blocs of surfaces that contain zeros of type I of orders $2a_1, \ldots, 2a_p$. For this construction the first and last surface is also a cylinder. Arranging the blocs in a cyclic order and identifying cylinders we get in each case an admissible topological type of configuration for $\mathcal{H}(\alpha)$ whose complementary regions are cylinders and tori.

Suppose now that all d_i in $\mathcal{H}(\alpha)$ are even, so we can not have a zero of type II. We then either have only zeros of type I or only zeros of type III as having zeros of types I and III necessarily implies that we have a zero of type II.

It is easy to verify (§14.1 in [EMZ]) that if we only have zeros of type I then the corresponding surface has an odd parity of spin structure and if we only only have zeros of type III then the parity of the resulting surface is the parity of $g-1$. So for g odd we are done, but for even g the parity of the spin structure is 1 in both cases.

If we only have zeros of type III but replace one of the pair of holes boundary tori with a genus 2 surface with a pair of holes boundary then the parity of the spin structure of the resulting surface is the parity of g. So for even g we constructed a surface of parity 0. This completes the proof. □

2.4 Toolbox

We recall some well known facts about the Beta function and incomplete Beta function.

The (real) *Gamma function* is defined for each $x > 0$ by

$$\Gamma(x) = \int_0^\infty e^{-u} u^{x-1} \mathrm{d}u.$$

It satisfies

$$\Gamma(x) = (x-1)\Gamma(x-1), \quad \text{so for } n \in \mathbb{N}, \quad \Gamma(n) = (n-1)!. \tag{2.9}$$

The *Beta function* defined for real numbers $a, b > 0$ by

$$\mathrm{B}(a,b) = \int_0^1 u^{a-1}(1-u)^{b-1} \mathrm{d}u \tag{2.10}$$

satisfies for positive real numbers a, b and positive integers n, m

$$\mathrm{B}(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \qquad \mathrm{B}(n,m) = \frac{(n-1)!(m-1)!}{(n+m-1)!}. \tag{2.11}$$

The *Incomplete Beta function* defined for real positive numbers x, a, b by

$$\mathrm{B}(x;a,b) = \int_0^x u^{a-1}(1-u)^{b-1} \mathrm{d}u \tag{2.12}$$

satisfies

$$\mathrm{B}(x;a,b) = \mathrm{B}(a,b) \sum_{k=a}^{a+b-1} \binom{a+b-1}{k} x^k (1-x)^{a+b-1-k}.$$

The *regularized incomplete Beta function* is defined as

$$\mathrm{I}(x;a,b) = \frac{\mathrm{B}(x;a,b)}{\mathrm{B}(1;a,b)} = \frac{\mathrm{B}(x;a,b)}{\mathrm{B}(a,b)}.$$

See figure 2.9 for the density function $f(x) = \frac{d}{dx}\,\mathrm{I}(x;a,b)$ for various values for a and b. See [Du] for a historical development of the incomplete Beta function.

Lemma 6.

$$I(A,B) = \sum_{k=0}^{B} (-1)^k \binom{A+B}{A+k} = \binom{A+B-1}{B} \tag{2.13}$$

$$\tilde{I}(A,B) = \sum_{k=0}^{B} k(-1)^{k+1} \binom{A+B}{A+k} = \binom{A+B-2}{B-1} \tag{2.14}$$

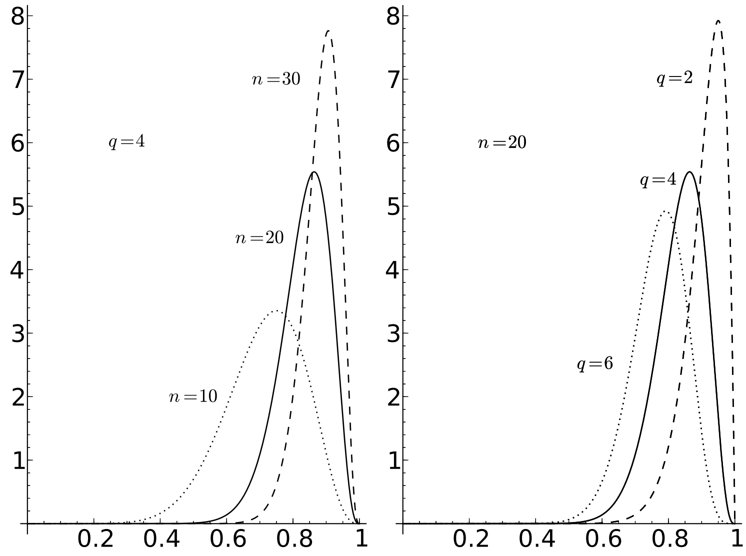


Figure 2.9: Graphs of the density function $f(x) = \frac{d}{dx}\,\mathrm{I}(x;a,b)$

Proof. Let $I_p(A,B) = \sum_{k=0}^{B}(-p)^k\binom{A+B}{A+k}$ and $\tilde{I}_p(A,B) = \sum_{k=0}^{B} k(-p)^k\binom{A+B}{A+k}$.

We have the recurrence relation $I_p(A,B+1) - (1-p)I_p(A,B) = \binom{A+B}{A-1}$. Taking $p = 1$ we get (2.13).

Taking the derivative of the recurrence relation with respect to p we get $\tilde{I}_p(A,B) = pI'_p(A,B)$ and hence $\tilde{I}_p(A,B+1) = -pI_p(A,B) + (1-p)\tilde{I}_p(A,B)$. Taking $p = 1$ we get (2.14). □

Lemma 7. *For $q \geq 0$ and $l \leq q$ the value of*

$$\hat{I}(n,q,l) = \sum_{k=0}^{q+1}\binom{n+q+1}{n+k}\binom{k}{q+1-l}(-1)^{l+k+q+1}$$

is

$$\hat{I}(n,q,l) = \binom{n+l-1}{l}.$$

Proof. Using the recurrence relation on binomial coefficients one obtains $\hat{I}(n,q+1,l) = \hat{I}(n,q,l)$. So we have $\hat{I}(n,q+1,l) = \hat{I}(n,l,l)$. Noting that $\hat{I}(n,l,l) = \tilde{I}(n,l+1)$ we get the desired result. □

Lemma 8. *The incomplete Beta function satisfies*

$$\mathrm{B}(1-p,n,q) = (1-p)^n\,\mathrm{B}(n,q)\sum_{l=0}^{q-1}\binom{n+l-1}{l}p^l.$$

Proof.

$$\begin{aligned}
&\sum_{k=n}^{n+q-1}\binom{n+q-1}{k}(1-p)^k p^{n+q-1-k}\\
&= (1-p)^n\sum_{k=0}^{q-1}\binom{n+q-1}{n+k}(1-p)^{n+k}p^{q-1-k}\\
&= (1-p)^n\sum_{k=0}^{q-1}\sum_{j=0}^{k}\binom{n+q-1}{n+k}\binom{k}{j}(-1)^{k-j}p^{q-1-j}\\
&= (1-p)^n\sum_{l=0}^{q-1}\left[\sum_{k=0}^{q-1}\binom{n+q-1}{n+k}\binom{k}{q-1-l}(-1)^{k+l+q+1}\right]p^l
\end{aligned}$$

It suffices to apply Lemma 7. $\square$

Chapter 3

Siegel–Veech constants for configurations of $\hat{h}$omologous saddle connections

This chapter corresponds to the first part of a preprint titled
“Siegel–Veech constants and volumes of strata of moduli spaces of quadratic differentials”.

3.1 Introduction

3.1.1 Cylinders and saddle connections on half-translation surfaces

A meromorphic quadratic differential q with at most simple poles on a Riemann surface S of genus g defines a flat metric on S with conical singularities. If q is *not* the global square of a holomorphic 1-form on S, the metric has a non-trivial linear holonomy group, and in this case (S, q) is called a *half-translation* surface. In this paper we consider only quadratic differentials satisfying the previous condition. If $\alpha = \{\alpha_1, \ldots, \alpha_n\} \subset \{-1\} \cup \mathbb{N}$ is a partition of $4g-4$, $\mathcal{Q}(\alpha)$ denotes the moduli space of pairs (S, q) as above, where q has exactly n singularities of orders given by α. It is a *stratum* in the moduli space $\mathcal{Q}_g$ of pairs (S, q) with no additional constraints on q.

In the following we will refer to a half-translation surface (S, q) simply as S.

A *saddle connection* on S is a geodesic segment on S joining a pair of conical singularities or a singularity to itself without any singularities in its interior. Note that maximal flat cylinders filled by parallel regular closed geodesics have their boundaries composed by one or several parallel saddle connections. In this paper we will evaluate the number of such cylinders on S in terms of the volumes of some strata, using the study of saddle connections by Masur and Zorich in [MZ].

3.1.2 Rigid collections of saddle connections

A saddle connection persists under any small deformation of S inside the stratum $\mathcal{Q}(\alpha)$. Moreover Masur and Zorich noticed in [MZ] that in some cases any small deformation which shortens a specific saddle connection shortens also some other saddle connections. More precisely, they give the following result (Proposition 1 of [MZ]):

Proposition 4 (Masur-Zorich)**.** *Let $\{\gamma_1, \ldots, \gamma_m\}$ be a collection of saddle connections on a half-translation surface S. Then any sufficiently small deformation of S inside the stratum preserves the proportions $|\gamma_1| : |\gamma_2| : \ldots : |\gamma_m|$ of the lengths of the saddle connections if and only if the saddle connections are $\hat{h}$omologous.*

Roughly two saddle connections are $\hat{h}$omologous if they define the same anti-invariant cycle in the orientation double cover. The precise definition will be recalled in § 3.2.1. In particular two $\hat{h}$omologous saddle connections are parallel with ratios of lengths equal to 1 or 2.

The geometric types of possible maximal collections of $\hat{\text{h}}$omologous saddle connections $\gamma = \{\gamma_1, \dots, \gamma_m\}$ on S are called *configurations* of saddle connections. Masur and Zorich classified all configurations of saddle connections in [MZ] in terms of combinatorial data.

We assume in the sequel that S belongs to a connected stratum (unless the non connectedness is stated explicitly), and we will speak indifferently about configurations for the surface S or for the stratum $\mathcal{Q}(\alpha)$, the second means that we look at all possible configurations on almost every surface $S \in \mathcal{Q}(\alpha)$.

We are interested in collections of $\hat{\text{h}}$omologous saddle connections, such that some of the saddle connections bound at least one cylinder filled by parallel regular closed geodesics. We refer to the geometric type of these collections as "configurations containing cylinders" or "configurations with cylinders".

It is proved in [MZ] that such cylinders have in fact each of their two boundaries composed by exactly one or two saddle connections in the collection, and that if there are several cylinders in the configuration, the lengths of their waist curves are either the same or have the ratio 1:2. Namely, some cylinders have their width twice larger than the width of the other cylinders. The boundary of the first cylinders are composed either by two or one saddle connection, and the boundary of the seconds are composed by exactly one saddle connection. We will refer to cylinders of the first type as "thick cylinders" and to cylinders of the second type as "thin cylinders". We call the length of the minimal saddle connection in the collection or equivalently the width of any thin cylinder the "length of the configuration".

Let γ be a maximal collection of $\hat{\text{h}}$omologous saddle connections on S. Then the complimentary region of these saddle connections and the cylinders bounded by these saddle connections is the union of some surfaces with boundaries. Each of them might be obtained by a specific surgery from a flat surface belonging to a stratum $\mathcal{Q}(\alpha_i)$ or $\mathcal{H}(\beta_j)$. The union of these strata $\mathcal{Q}(\alpha') = \cup_{i,j} \mathcal{Q}(\alpha_i) \cup \mathcal{H}(\beta_j)$ is called the ***boundary stratum*** for the configuration $\mathcal{C}$. This denomination is meaningful: the boundary stratum corresponds to the degeneration of the stratum $\mathcal{Q}(\alpha)$ as the lengths of the saddle connections in the collection tend to 0.

3.1.3 Counting saddle connections

Let S be a half-translation surface in a connected stratum $\mathcal{Q}(\alpha)$, and $\mathcal{C}$ a configuration with cylinders on S. It means that in some given direction, there is a collection of $\hat{\text{h}}$omologous saddle connections of type $\mathcal{C}$ on S. Note that by results of [EMa] in many other directions, one can usually find another collection of $\hat{\text{h}}$omologous saddle connections of same type $\mathcal{C}$.

We introduce $N(S, \mathcal{C}, L)$ the number of directions on S in which we can find a collection of saddle connections of type $\mathcal{C}$, with the length of the smallest saddle connection smaller than L. Since we are interested in cylinders we introduce also $N_{cyl}(S, \mathcal{C}, L)$ that counts each appearance of the configuration $\mathcal{C}$ with weight equal to the number of the cylinders of width smaller than L, and $N_{area}(S, \mathcal{C}, L)$ that counts each appearance of the configuration $\mathcal{C}$ with weight equal to the area of the cylinders of width smaller than L.

For each of these numbers, we introduce the corresponding Siegel–Veech constant, that gives the asymptotic of these numbers as L goes to infinity:

$$c_*(\mathcal{C}) = \lim_{L \to \infty} \frac{N_*(S, \mathcal{C}, L) \cdot (\text{Area of } S)}{\pi L^2}$$

Eskin and Masur showed in [EMa] that these constants do not depend on S for almost every S in the connected stratum $\mathcal{Q}(\alpha)$. Combining these results with the results of Veech ([Ve3]), one concludes that all these constants are strictly positive.

3.1.4 Application of Siegel–Veech constants

One of the principal reasons, why the Siegel–Veech constants are more and more intensively studied during the last years (see [AEZ1], [Ba1], [Ba2], [BG], [EKZ2], [Vo]) is the relation between them and the Lyapunov exponents of the Hodge bundle along the Teichmüller flow: the key formula of [EKZ2] expresses the sum of the positive Lyapunov exponents for any stratum $\mathcal{Q}(\alpha)$ as a sum of a very explicit rational function in α and the Siegel–Veech constant $c_{area}(\mathcal{Q}(\alpha))$. The Lyapunov exponents are closely related to the deviation spectrum of measured foliations on individual flat surfaces, see [Fo1], [Fo2], [Zo1], [Zo2], which opens applications to billiards in polygons, interval exchanges, etc.

A recent breakthrough of A. Eskin and M. Mirzakhani provides, in particular, new tools allowing to prove that the $\mathrm{SL}(2,\mathbb{R})$-orbit closure of certain individual flat surfaces is an entire stratum. By the theorem of J. Chaika and A. Eskin [CkE], almost all directions for such a flat surface are Lyapunov-generic. This allows to cumulate all the technology mentioned above to compute, for example, the diffusion rate of billiards with certain periodic obstacles. The final explicit answer (as 2/3 for the diffusion rate in the windtree model studied in [DHL]) is certain Lyapunov exponent as above. This kind of quantitative answers or estimates are often reduced to computation of the appropriate Siegel–Veech constant.

The Konsevich formula (see [Ko2]) for the sum of the Lyapunov exponents over a Teichmüller curve and recent results of S. Filip [Fi] showing that every orbit closure is a quasiprojective variety suggest that an adequate intersection theory of the strata might provide algebro-geometric tools to evaluate Siegel–Veech constants (see also [KtZg] in this connection). However, such intersection theory is not developed yet, and we are limited to analytic tools in our evaluation of Siegel–Veech constants.

3.1.5 Principal results

Now we are ready to state the main theorem of this paper.

Theorem 8. *Let $\mathcal{C}$ be an admissible configuration for a connected stratum $\mathcal{Q}(\alpha)$ of quadratic differentials. Let q_1 denote the number of thin cylinders, q_2 the number of thick cylinders in the configuration $\mathcal{C}$, and $q = q_1 + q_2$ the total number of cylinders. Assume that the boundary stratum $\mathcal{Q}(\alpha')$ is non empty, and $q \geq 1$. Then the Siegel–Veech constants associated to $\mathcal{C}$ are the following:*

$$c(\mathcal{C}) = \frac{M}{2^{q+2}} \frac{(\dim_{\mathbb{C}} \mathcal{Q}(\alpha') - 1)!}{(\dim_{\mathbb{C}} \mathcal{Q}(\alpha) - 2)!} \frac{\mathrm{Vol}\, \mathcal{Q}_1(\alpha')}{\mathrm{Vol}\, \mathcal{Q}_1(\alpha)} \tag{3.1}$$

$$c_{cyl}(\mathcal{C}) = \left(q_1 + \frac{1}{4} q_2\right) c(\mathcal{C}) \tag{3.2}$$

$$c_{area}(\mathcal{C}) = \frac{1}{\dim_{\mathbb{C}} \mathcal{Q}(\alpha) - 1} c_{cyl}(\mathcal{C}) \tag{3.3}$$

where $M = \dfrac{M_s M_c}{M_t}$ and M_c, M_t, M_s are combinatorial constants depending only on the configuration $\mathcal{C}$, explicitly given by equations (3.8), (3.11) and (3.18).

When the boundary stratum is empty, the formulae are simpler and given in §3.3.3.

This theorem is proven in section 3.3.3. Note that these formulae coincide in genus 0 with formulae of [AEZ1], for the two configurations containing cylinders (named "pocket" and "dumbell" in the article).

The ratio $\dfrac{c_{area}(\mathcal{C})}{c_{cyl}(\mathcal{C})} = \dfrac{1}{\dim_{\mathbb{C}} \mathcal{Q}(\alpha) - 1}$ can be interpreted as the mean area of a cylinder in the configuration $\mathcal{C}$. Note that it depends only on the dimension of the ambient stratum.

For a fixed stratum $\mathcal{Q}(\alpha)$ consider all admissible configurations, and denote $q_{max}(\alpha)$ the maximal number of cylinders for all these configurations. We evaluate this number in section 3.5.2. The ratio $\dfrac{q_{max}(\alpha)}{\dim_{\mathbb{C}}(\mathcal{Q}(\alpha)) - 1}$ represents the maximum mean total area of the cylinders in stratum $\mathcal{Q}(\alpha)$.

Proposition 5. *We have*

$$\max_{\alpha \in \Pi(4g-4+k)} \frac{q_{\max}(\alpha \cup \{-1^k\})}{2g - 3 + \ell(\alpha) + k} \xrightarrow[g\to\infty]{k\,fixed} \frac{1}{3}$$
$$\xrightarrow[k\to\infty]{g\,fixed} \frac{1}{5}$$

where $\Pi(4g-4+k)$ *denotes the set of partitions of* $4g-4+k$ *and* $l(\alpha)$ *is the length of the partition* α*. Furthermore for any genus* g *and number of poles* k *the bound is achieved for* $\alpha \in \Pi k' \sqcup \Pi_4(4g-4+k-k')$*, where* $k' = k - 4\lfloor\frac{k}{4}\rfloor$ *and* $\Pi_4(4g-4+k-k')$ *denote the set of partitions of* $4g-4+k-k'$ *using only* 4*'s.*

3.1.6 Historical remarks

The Siegel–Veech constants for the strata of Abelian differentials were evaluated in the paper [EMZ]; the relations between various Siegel–Veech constants were studied in [Vo] and some further ones in a recent paper [BG]. The computation in [EMZ] involves a combination of rather involved combinatorial and geometric constructions. To test the consistence of their theoretical predictions numerically, the authors of [EMZ] compare the formulae for the Lyapunov exponents expressed in terms of the Siegel–Veech constants (reduced, in turn, to combinations of volumes of the boundary strata) with numerics provided by experiments with the Lyapunov exponents. These tests are based, in particular, on the results of A. Eskin and A. Okounkov [EOk1] providing the explicit values of the volumes of all strata of Abelian differentials in small genera.

The description of combinatorial geometry of configurations of saddle connections for the strata of quadratic differentials is performed in the paper of H. Masur and A. Zorich [MZ]; for the hyperelliptic components and for strata in genus zero such description is given in the paper of C. Boissy [Bo].

The evaluation of the corresponding Siegel–Veech constants in genus zero was recently performed by J. Athreya, A. Eskin, and A. Zorich [AEZ1]; see also the related paper [AEZ2]. The results were also verified by computer experiments with Lyapunov exponents combined with the knowledge of the volumes of the strata of quadratic differentials in genus zero. (The authors prove in [AEZ1] an extremely simple explicit formula for such volumes in genus zero conjectured by M. Kontsevich.)

In the current paper we treat the strata of quadratic differentials in arbitrary genus. We should point out that we are currently very limited in numerical tests of the suggested formulae. In the contrast to the strata of Abelian differentials the analogous results of A. Eskin, A. Okounkov, and R. Pandharipande [EOPa] do not provide explicit values for the volumes of the strata of quadratic differentials. This is why we have included in this paper a straightforward evaluation of volumes of certain strata, which allows to obtain at least some exact values of Siegel–Veech constants for the strata of quadratic differentials away from genus zero, and to show that our formulae for Siegel–Veech constants are consistent with numerics coming from Lyapunov exponents of the Hodge bundle over the Teichmüller flow.

3.1.7 Structure of the paper

The paper is divided into two parts. The first part, theoretical, gives the proof of Theorem 8, and develops the results on a special family of strata: $\mathcal{Q}(1^k, -1^l)$. The first part ends with the extension of some geometric results proved in [BG] for the strata of Abelian differentials to the strata of quadratic differentials.

The computations of this first part generalize the computations presented in the articles [EMZ], and [AEZ1], but in higher genus there is a huge distance between the theory and getting exact values of Siegel–Veech constants, because the techniques involve some phenomenons of higher complexity. This is why we present in a second part all pragmatical computations.

So the second part of this paper is devoted to the computation of the values of the volumes of certain strata and of the hyperelliptic components of strata. Since the values of the corresponding Siegel–Veech constants are known, the computed values of volumes enable us to check that the formulae of the first part are coherent. This checking is primordial since the choice of the normalization for the volume and the symmetries of high complexity for the configurations affect each step of the computations. In the Abelian case this checking has been done using numerical values provided by [EOk1], but for the case of quadratic differentials obtaining numerical values of volumes is still work in progress. So our hope is also that the computed values of this article will be used as test values for the future effective algorithm giving approximated values of volumes of strata of quadratic differentials.

In this part we first recall the prerequisites and the general method of [EMZ] to compute Siegel–Veech constants. In section 3.3 we prove Theorem 8. Then we apply our results in § 3.4 to the family of strata $\mathcal{Q}(1^k, -1^l)$. In this case we can give very explicitly the constant M which appears in Theorem 8. Finally in the last section of this part we extend some geometric results proven in [BG] in the case of Abelian differentials.

3.2 Preliminaries

3.2.1 $\hat{\text{H}}$omologous saddle connections

We precise here from [MZ] the notion of *$\hat{h}$omologous* saddle connections.

Recall that any flat surface (S, q) in $\mathcal{Q}(\alpha)$ admits a canonical ramified double cover $\hat{S} \xrightarrow{p} S$ such that the induced quadratic differential on $\hat{S}$ is a global square of an Abelian differential, that is $p^*q = \omega^2$ and $(\hat{S}, \omega) \in \mathcal{H}(\hat{\alpha})$. Let $\Sigma = \{P_1, \dots P_n\}$ denote the singular points of the quadratic differential on S, and $\hat{\Sigma} = \{\hat{P}_1, \dots \hat{P}_N\}$ the singular points of the Abelian differential ω on $\hat{S}$. Note that the pre-images of poles P_i are regular points of ω so do not appear in the list $\hat{\Sigma}$. The subspace $H^1_-(\hat{S}, \hat{\Sigma}; \mathbb{C})$ antiinvariant with respect to the action of the hyperelliptic involution provides local coordinates in the stratum $\mathcal{Q}(\alpha)$ in the neighbourhood of S.

Let γ be a saddle connection on S. We denote γ' and γ'' its two lifts on $\hat{S}$. If $[\gamma] = 0$ downstairs, then $[\gamma'] + [\gamma''] = 0$ upstairs, and in this case we define $[\hat{\gamma}] := [\gamma']$. In the other case we have $[\gamma'] + [\gamma''] \neq 0$ and we define $[\hat{\gamma}] := [\gamma'] - [\gamma'']$. We obtain an element of $H^1_-(\hat{S}, \hat{\Sigma}; \mathbb{C})$.

Then two saddle connections γ_1 and γ_2 are said to be $\hat{\text{h}}$omologous if $[\hat{\gamma}_1] = [\hat{\gamma}_2]$ in $H_1(\hat{S}, \hat{\Sigma}, \mathbb{Z})$, under an appropriate choice of orientations of γ_1, γ_2.

3.2.2 Configurations of saddle connections

A configuration is one of the geometric type of all possible maximal collections of $\hat{\text{h}}$omologous saddle connections. We explain here precisely which informations characterize the geometric type of a collection (Definition 3 of [MZ]). Given such a collection of saddle connections on a surface S, cutting along these saddle connections will give a union of surfaces with boundaries. These surfaces can be either flat cylinders, or surfaces obtained by a surgery from a surface of trivial or non trivial holonomy. These surfaces are called boundary surfaces. We record the genus and the order of the singularities of all these surfaces. We record also which type of surgery is applied to which singularity on each surface with the precise angles. Finally we record the way the surfaces are glued in the initial surface. All this information characterizes a configuration of $\hat{\text{h}}$omologous saddle connections.

3.2.3 Graphs of configurations

We recall here briefly how the graphs introduced by Masur and Zorich in [MZ] encode all combinatorial information about a configuration. Let S be a half-translation and γ a saddle connection of configuration $\mathcal{C}$. The graph of the configuration $\mathcal{C}$ is given by the following procedure: associate to each of the boundary surfaces a vertex in the graph, with the following symbolic: a vertex $\oplus$ represents a surface of trivial holonomy, a vertex $\ominus$ a surface of non trivial holonomy, and a vertex $\circ$ a cylinder. Then there is an edge between two vertices if the boundaries of the corresponding surfaces share a common saddle connection. At this stage we obtain a graph described by Figure 3 in [MZ].

The surgeries performed on each surface are represented by local ribbon graphs belonging to the list described in Figure 6 of [MZ]. These local graphs are decorated with numbers k_i which are the numbers of horizontal geodesic rays emerging from the zeroes on which we perform the surgery, in an angular sector delimited by two $\hat{\text{h}}$omologous saddle connections. The reunion of these local ribbon graphs forms globally a ribbon graph that can be drawn on the graph giving the organization of the surfaces. The boundary of this ribbon graph has several connected components, each of them represents a newborn zero. To compute the order of a newborn zero, one can count the number of geodesic rays emerging from this point, that is, sum all the k_i's met when one goes along the connected component of the boundary of the ribbon graph corresponding to the newborn zero. The cone angle around this point is then $\pi\sum_i(k_i+1)$. See Figure 7 in [MZ] for an example.

3.2.4 General strategy for the computation of Siegel–Veech constants

We recall here the sketch of the general method developed in [EMZ] to evaluate Siegel–Veech constants in the Abelian case, transposed to the quadratic case in genus 0 in [AEZ1].

Let $V_{\mathcal{C}}(S)$ be the set of holonomy vectors of saddle connections on S of type $\mathcal{C}$. The number of configurations $\mathcal{C}$ in S such that the length of the $\hat{\text{h}}$omologous saddle connections is bounded is then

$$N(S,\mathcal{C},L)=\frac{1}{2}|V_{\mathcal{C}}(S)\cap B(0,L)|,$$

where the factor $\frac{1}{2}$ compensates the fact that the saddle connections are not oriented and so their holonomy vectors are defined up to a sign. If q is the number of cylinders in the configuration and q_1 the number of "thin" cylinders, we define as well

$$N_{cyl}(S,\mathcal{C},L)=\frac{1}{2}\left(q\left|V_{\mathcal{C}}(S)\cap B\left(0,\frac{L}{2}\right)\right|+q_1\left|V_{\mathcal{C}}(S)\cap A\left(\frac{L}{2},L\right)\right|\right),$$

with $A\left(\frac{L}{2},L\right)=B(0,L)\setminus B\left(0,\frac{L}{2}\right)$. Note that $N_{cyl}(S,\mathcal{C},L)$ counts each realization of configuration $\mathcal{C}$ with weight the number of cylinders of width smaller than L: if the width of the thin cylinders is smaller than $L/2$ then all the q cylinders have their width smaller than L, if the width of the thin cylinders is comprised between $L/2$ and L, then the thick cylinders do not count.

Simplifying the last expression we get

$$N_{cyl}(S,\mathcal{C},L)=q_2N(S,\mathcal{C},L/2)+q_1N(S,\mathcal{C},L) \tag{3.4}$$

where q_2 is the number of thick cylinders ($q=q_1+q_2$).

Finally we define

$$N_{area}(S,\mathcal{C},L)=\frac{1}{2}\sum_{v\in V_{\mathcal{C}}(S)\cap B(0,L)}A(v)$$

where $A(v)$ is the area of the cylinders of width smaller than L among those associated to the saddle connections of type $\mathcal{C}$ and holonomy vector $\pm v$. Note that $N_{area}(S,\mathcal{C},L)$ weights only the cylinders which are counted by $N_{cyl}(S,\mathcal{C},L)$.

Convention 1. Following [AEZ1] we denote $\mathcal{Q}_1(\alpha)$ the hypersurface in $\mathcal{Q}(\alpha)$ of flat surfaces of area 1/2 such that the area of the double cover is 1.

The stratum $\mathcal{Q}(\alpha)$ is equipped with a natural $PSL(2,\mathbb{R})$-invariant measure μ, called Masur-Veech measure, induced by the Lebesgue measure in period coordinates. We choose a normalization for μ in 3.3.1. This measure induces a measure μ_1 on $\mathcal{Q}_1(\alpha)$ in the following way: if E is a subset of $\mathcal{Q}_1(\alpha)$, we denote $C(E)$ the cone underneath E in the stratum $\mathcal{Q}(\alpha)$:

$$C(E)=\{S\in\mathcal{Q}(\alpha)\text{ s.t. }\exists r\in(0,+\infty),S=rS_1\text{ with }S_1\in E\}$$

and we define

$$\mu_1(E)=2d\cdot\mu(C(E)),$$

with $d=\dim_C\mathcal{Q}(\alpha)$, that is, the measure $\mathrm{d}\mu$ disintegrates in $\mathrm{d}\mu=r^{2d-1}\mathrm{d}r\mathrm{d}\mu_1$.

Eskin and Masur proved in [EMa] that the asymptotics

$$\lim_{L\to\infty}\frac{N_*(S,\mathcal{C},L)\cdot(\text{Area of }S)}{\pi L^2}$$

do not depend on the surface S for almost every surface in a connected component of a stratum of Abelian differentials. Athreya Eskin and Zorich generalized their method to the quadratic case in Theorem 2.3 in [AEZ1]. Then the following Siegel–Veech constants are well defined for almost every surface S in a connected component of a stratum of quadratic differentials:

$$c_*(\mathcal{C}) = \lim_{L\to\infty} \frac{N_*(S,\mathcal{C},L)\cdot(\text{Area of } S)}{\pi L^2}.$$

Remark 1. Note that it follows directly from this formula and the definition (3.4) of $N_{cyl}(S,\mathcal{C},L)$ that:

$$c_{cyl}(\mathcal{C}) = \left(q_1 + \frac{1}{4}q_2\right) c(\mathcal{C}),$$

which is the equation (3.2) in Theorem 8.

Now let $\mathcal{Q}(\alpha)$ be connected stratum. The Siegel–Veech formula (cf [Ve3], Theorem 0.5) gives the existence of constants $b_*(\mathcal{C})$ such that

$$\frac{1}{\mathrm{Vol}(\mathcal{Q}_1(\alpha))}\int_{\mathcal{Q}_1(\alpha)} N_*(S,\mathcal{C},L)\mathrm{d}\mu_1(S) = b_*(\mathcal{C})\pi L^2$$

so necessarily $b_*(\mathcal{C}) = 2c_*(\mathcal{C})$ and we can express the Siegel–Veech constant as

$$c_*(\mathcal{C}) = \lim_{\varepsilon\to 0}\frac{1}{2\pi\varepsilon^2}\frac{1}{\mathrm{Vol}(\mathcal{Q}_1(\alpha))}\int_{\mathcal{Q}_1(\alpha)} N_*(S,\mathcal{C},\varepsilon)\mathrm{d}\mu_1(S).$$

In fact the integral is over the subset $\mathcal{Q}_1^\varepsilon(\mathcal{C})$ of $\mathcal{Q}_1(\alpha)$ formed by the surfaces with at least one family of "short" saddle connections of type $\mathcal{C}$, where "short" means of length smaller than ε. We decompose this subset as $\mathcal{Q}_1^\varepsilon(\mathcal{C}) = \mathcal{Q}_1^{\varepsilon,thick}(\mathcal{C}) \cup \mathcal{Q}_1^{\varepsilon,thin}(\mathcal{C})$ where $\mathcal{Q}_1^{\varepsilon,thin}(\mathcal{C})$ is the set of surfaces having at least two distinct collections of short saddle connections of type $\mathcal{C}$. Eskin and Masur proved that this subset is so small that we have

$$\frac{1}{\mathrm{Vol}\,\mathcal{Q}_1(\alpha)}\int_{\mathcal{Q}_1^{\varepsilon,thin}(\mathcal{C})} N_*(S,\mathcal{C},\varepsilon)\mathrm{d}\mu_1(S) = o(\varepsilon^2).$$

Finally we obtain

$$c_*(\mathcal{C}) = \lim_{\varepsilon\to 0}\frac{1}{2\pi\varepsilon^2}\frac{\mathrm{Vol}_*\,\mathcal{Q}_1^\varepsilon(\mathcal{C})}{\mathrm{Vol}\,\mathcal{Q}_1(\alpha)} \tag{3.5}$$

where $\mathrm{Vol}_*\,\mathcal{Q}_1^\varepsilon(\mathcal{C})$ is the weighted volume:

$$\mathrm{Vol}_*\,\mathcal{Q}_1^\varepsilon(\mathcal{C}) = \int_{\mathcal{Q}_1^\varepsilon(\mathcal{C})} W_*(\mathcal{C},S)\mathrm{d}\mu_1(S)$$

with $W(\mathcal{C},S) = 1$, $W_{cyl}(\mathcal{C},S)$ is equal to the number of cylinders of width smaller than ε, $W_{area}(\mathcal{C},S)$ is equal to the area of the cylinders of length smaller than ε in the configuration $\mathcal{C}$ on S.

The last step is the computation of $\mathrm{Vol}_*\,\mathcal{Q}_1^\varepsilon(\mathcal{C})$ in term of the volume of the boundary stratum, see § 3.3.3.

Counting saddle connections of type $\mathcal{C}$ is related to a more general problem: counting saddle connections with no fixed type. Introducing the number $N(S,L)$ of distinct holonomies of saddle connections shorter than L on $S\in\mathcal{Q}(\alpha)$, the corresponding Siegel–Veech constants

$$c_*(\mathcal{Q}(\alpha)) = \lim_{L\to\infty}\frac{N(S,L)\cdot(\text{Area of } S)}{\pi L^2}$$

are also well-defined for almost every $S\in\mathcal{Q}(\alpha)$ and depend only of the stratum. Then we have naturally

$$c_*(\mathcal{Q}(\alpha)) = \sum_{\mathcal{C}} c_*(\mathcal{C}).$$

The constant $c_{area}(\mathcal{Q}(\alpha))$ is particularly important because the formula of [EKZ2] relates it to the sum of Lyapunov exponents for the Teichmüller geodesic flow. So it implies a lot a applications to the dynamics in polygonal billiards. Also since there are numerical experiments on Lyapunov exponents, the Eskin-Kontsevitch-Zorich formula provides numerical approximation for the constants $c_{area}(\mathcal{Q}(\alpha))$, and that gives a way to check computations on the constants $c_{area}(\mathcal{C})$. This is the main reason why we focus on configurations containing cylinders: they are the only ones that contribute to the constant $c_{area}(\mathcal{Q}(\alpha))$.

3.2.5 Strata that are not connected

In the last section we explained the method to compute Siegel–Veech constants for connected strata. The classification of connected components of strata is given in [La2]. Most of the strata are connected, the only ones which are not connected are the one which have a hyperelliptic component (except some sporadic examples in genus 3 and 4), and in this case there is only one supplementary component. The three types of strata containing hyperelliptic components are recalled on § 4.1.1.

The general strategy for computing Siegel–Veech constants for the connected strata can be adapted for connected components. For a connected component $\mathcal{Q}^{comp}(\alpha)$ we define the Siegel–Veech constants by the means:

$$c_*(\mathcal{Q}^{comp}(\alpha), \mathcal{C}) = \lim_{\varepsilon \to 0} \frac{1}{2\pi\varepsilon^2} \frac{1}{\mathrm{Vol}(\mathcal{Q}_1^{comp}(\alpha))} \int_{\mathcal{Q}_1^{comp}(\alpha)} N(S, \mathcal{C}, \varepsilon) \mathrm{d}\mu_1(S).$$

Note that the connected components of $\mathcal{Q}_1(\alpha)$ are exactly the intersection of $\mathcal{Q}_1(\alpha)$ with the connected components of $\mathcal{Q}(\alpha)$. We have also the property that

$$c_*(\mathcal{C}) = \lim_{L \to \infty} \frac{N_*(S, \mathcal{C}, L) \cdot (\text{Area of } S)}{\pi L^2},$$

for almost every S in the component $\mathcal{Q}^{comp}(\alpha)$.

So we will obtain the same evaluation:

$$c_*(\mathcal{Q}^{comp}(\alpha), \mathcal{C}) = \lim_{\varepsilon \to 0} \frac{1}{2\pi\varepsilon^2} \frac{\mathrm{Vol}_* \, \mathcal{Q}_1^{\varepsilon}(comp, \mathcal{C})}{\mathrm{Vol}\, \mathcal{Q}_1(\alpha)}. \tag{3.6}$$

We apply this method in the case of hyperelliptic components in section 4.2.

3.3 Computation of Siegel-Veech constant for connected strata

In this section, $\mathcal{Q}(\alpha)$ will denote a connected stratum of quadratic differentials. We will evaluate Siegel–Veech constants $c_*(\mathcal{C})$ defined in § 3.2.4 using equation (3.5).

3.3.1 Choice of normalization

We have to choose a normalization for the volume element on a strata $\mathcal{Q}(\alpha)$, which is equivalent to choose a lattice in the space $H^1_-(\hat{S}, \hat{\Sigma}; \mathbb{C})$ which gives the local model of the stratum $\mathcal{Q}(\alpha)$ around S.

Convention 2. We follow the convention of [AEZ1] and choose, as lattice in $H^1_-(\hat{S}, \hat{\Sigma}; \mathbb{C})$ of covolume 1, the subset of those linear forms which take values in $\mathbb{Z} \oplus i\mathbb{Z}$ on $H_1^-(\hat{S}, \hat{\Sigma}; \mathbb{Z})$, that we will denote by $(H_1^-(\hat{S}, \hat{\Sigma}; \mathbb{Z}))^*_{\mathbb{C}}$.

This convention implies that the non zero cycles in $H_1(S, \Sigma, \mathbb{Z})$ (that is, those represented by saddle connections joining two distinct singularities or closed loops non homologous to zero) have half-integer holonomy, and the other ones (closed loops homologous to zero) have integer holonomy.

Convention 3. We choose to labelled all zeroes and poles. This affects the computation of volumes, but it is easy to deduce the value of volumes of strata with anonymous singularities.

3.3.2 Construction of a basis of $H_1^-(\hat{S}, \hat{\Sigma}, \mathbb{Z})$

In this section we recall the generic construction given in [AEZ1] of a basis of $H_1^-(\hat{S}, \hat{\Sigma}, \mathbb{Z})$ from a basis of $H_1(S, \Sigma, \mathbb{Z})$, and also a specific construction for each configuration. In the following sections we will look at every configuration and use the specific basis associated to each configuration in order to have a nice expression of the measure in terms of parameters of the cylinders.

For a primitive cycle $[\gamma]$ in $H_1(S, \Sigma, \mathbb{Z})$, that is, a saddle connection joining distinct zeros or a closed cycle (absolute cycle), the lift $[\hat{\gamma}]$ is a primitive element of $H_1^-(\hat{S}, \hat{\Sigma}, \mathbb{Z})$.

"Generic" basis

(cf [AEZ1] §3.1) Let k be the number of poles in Σ, a the number of even zeroes and b the number of odd zeroes (of order ≥ 1). Assume that the zeroes are numbered in the following way: $P_1, \dots P_a$ are the even zeroes, $P_{a+1}, \dots, P_{a+b}$ are the odd zeroes and $P_{a+b+1}, \dots, P_n$ the poles, and take a simple oriented broken line $P_1, \dots P_{n-1}$. Take each saddle connexion γ_i represented by $[P_i, P_{i+1}]$ for i going from 1 to $n-2$, and a basis $\{\gamma_{n-1}, \dots, \gamma_{n+2g-2}\}$ of $H_1(S, \mathbb{Z})$.

Lemma 3. *The family $\{\hat{\gamma}_1, \dots, \hat{\gamma}_{n+2g-2}\}$ is a basis of $H_1^-(\hat{S}, \hat{\Sigma}, \mathbb{Z})$.*

Proof. First it is clear that the elements $\hat{\gamma}_1, \dots, \hat{\gamma}_{n+2g-2}$ are primitive elements of $H_1^-(\hat{S}, \hat{\Sigma}, \mathbb{Z})$ and linearly independent. Moreover they do not generate a proper sub-lattice of $H_1^-(\hat{S}, \hat{\Sigma}, \mathbb{Z})$. Each of the k poles lifts to a regular point in $\hat{S}$ so does not appear in the list $\hat{\Sigma}$. An even zero of order α_i lifts to two zeroes of degrees $\frac{\alpha_i}{2}$, and an odd zero of order α_j lifts to a zero of degree $\alpha_j + 1$. So we have $n = |\Sigma| = k + a + b$ and $N = |\hat{\Sigma}| = 2a + b$. Thus if $\hat{g}$ is the genus of $\hat{S}$ we have $4g - 4 = -k + \sum_{\alpha_i \geq 1} \alpha_i$ and $2\hat{g} - 2 = \sum_{\alpha_i \geq 1} \alpha_i + b$ and so

$$\begin{aligned} \dim_{\mathbb{C}}(H_1(\hat{S}, \hat{\Sigma}, \mathbb{Z})) = 2\hat{g} - 1 + N &= (2g - 2 + n) + (2g - 1 + a + b) \\ &= \dim_{\mathbb{C}} H_1^-(\hat{S}, \hat{\Sigma}, \mathbb{C}) + \dim_{\mathbb{C}} H_1^+(\hat{S}, \hat{\Sigma}, \mathbb{C}). \end{aligned}$$

This equality on dimensions shows that we can complete the family $\{\hat{\gamma}_1, \dots, \hat{\gamma}_{n+2g-2}\}$ with $\{\gamma'_1, \dots, \gamma'_{n-k-1}, \gamma'_{n-1}, \dots, \gamma'_{n+2g-2}\}$ to form a basis of $H_1(\hat{S}, \hat{\Sigma}, \mathbb{R})$ (the linear independence is clear from the construction). The intersection matrix has integer coefficients and is of determinant 1, so that ends the proof of the lemma. □

Basis associated to a configuration

Fix a configuration $\mathcal{C}$. As in [EMZ], we define an appropriate family $\{\gamma_1, \dots, \gamma_{n+2g-2}\}$ of $H_1(S, \Sigma, \mathbb{Z})$ for $S \in \mathcal{C}$, which lifts to a basis of $H_1^-(\hat{S}, \hat{\Sigma}, \mathbb{Z})$, as follows:

- for each component of the principal boundary strata $\mathcal{Q}(\alpha'_i)$ take a family $\{\beta_1^i, \dots, \beta_{n_i+2g_i-2}^i\}$ of $H_1(S'_i, \Sigma_i, \mathbb{Z})$ such that $\{\hat{\beta}_1^i, \dots, \hat{\beta}_{n_i+2g_i-2}^i\}$ is a basis of $H_1^-(\hat{S}'_i, \hat{\Sigma}_i, \mathbb{Z})$ as previously,
- for each $\hat{\text{h}}$omologous cylinder take a curve δ_j joining its boundary singularities (there might be an ambiguity in the choice of such a curve, cf § 3.3.3)
- take a saddle connection or a closed curve in the homology class of γ (we denote $\pm\overrightarrow{v}$ the holonomy of γ).

Lifting this basis to $H_1^-(\hat{S}, \hat{\Sigma}, \mathbb{Z})$ using the $\hat{}$ operator provides a primitive basis of $H_1^-(\hat{S}, \hat{\Sigma}, \mathbb{Z})$, as previously.

We will keep the same notations for elements in $(H_1^-(\hat{S}, \hat{\Sigma}; \mathbb{Z}))^*_{\mathbb{C}}$

3.3.3 Computation

Fix a configuration $\mathcal{C}$ containing q cylinders ($q \geq 1$). Now we give a complete description of the measure μ in terms of parameters of the configuration by disintegrating the volume element $\mathrm{d}\mu$.

By [EMa] and [MS] we have $\mathrm{Vol}_* \mathcal{Q}_1^\varepsilon(\mathcal{C}) = \mathrm{Vol}_* \mathcal{Q}_1^{\varepsilon,thick}(\mathcal{C}) + o(\varepsilon^2)$, so we will describe μ only on $\mathcal{Q}^{\varepsilon,thick}(\mathcal{C})$.

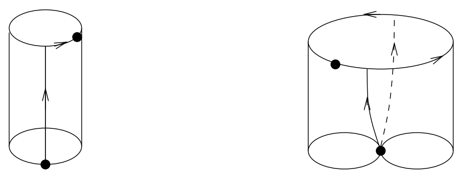
Let $S \in \mathcal{Q}^{\varepsilon,thick}(\mathcal{C})$. Local coordinates near S are given by $H_-^1(\hat{S}, \hat{\Sigma}, \mathbb{C})$, and μ is just Lebesgue measure in this coordinates. Choose now a basis associated to the configuration $\mathcal{C}$ as above. It follows from the papers [EMZ] and [MZ] that the measure $\mathrm{d}\mu$ in $\mathcal{Q}^{\varepsilon,thick}(\mathcal{C})$ disintegrates as the product of the measure $\mathrm{d}\mu'$ on $\mathcal{Q}(\alpha')$ and the measure $\mathrm{d}\nu_T$ on the space of parameters $\mathcal{T}$ of the cylinders:

$$\mathrm{d}\mu = M' \mathrm{d}\mu' \mathrm{d}\nu_T$$

where M' denotes the number of ways to get a surface S in $\mathcal{Q}^{\varepsilon,thick}(\mathcal{C})$ when the parameters of the configuration are fixed.

Description of the space $\mathcal{T}$ of the cylinders

Roughly $\mathcal{T}$ is described by coordinates $\pm\overrightarrow{v}, h_1, \dots, h_q, t_1, \dots t_q$ representing the width, the heights and the twists of the cylinders, defined such that $h_i + it_i$ is the holonomy of the curve δ_i. The problem here is that there might be an ambiguity for the choice of this curve and so for the definition of the twist. In the following we assume that the cylinders are horizontal, that is $\pm\overrightarrow{v}$ represents the horizontal direction in the surface S. First note that despite the fact that the surface have a non trivial holonomy, for a given configuration $\mathcal{C}$ it is possible to choose an orientation for each cylinder, for example by choosing an oriented path covering the graph representing the configuration. So in each cylinder we have a notion of bottom, up, left and right. Recall that thin cylinders are the one with each of their boundaries formed by a single saddle connection of holonomy $\pm\overrightarrow{v}$, and so there is only one singularity on each of their boundaries. For these cylinders we can define the twist and the height of the cylinder as usual: starting from the only one singularity on the bottom of the cylinder, draw a vertical segment going up and ending at a point P on the upper boundary of the cylinder. The length of this segment defines the height of the cylinder. Starting from the point P and following the boundary in the right horizontal direction, we meet the singularity on the upper boundary of the cylinder, which is at distance t from P, and t defines the twist of the cylinder ($0 \le t < |\overrightarrow{v}|$). The next picture shows a particular case where the twist is ambiguous for a thick cylinder.



no ambiguity ambiguity

For the thick cylinders, we can define their twist as follows: for such a cylinder, if one of its boundaries contains two distinct singularities (recall that the singularities are labelled), then choose the one of the smaller index. We have now in each case one distinguished singularity on each of the two boundaries. Consider the shortest geodesic segments joining these two singularities (there might be two such segments). Then their vertical coordinates coincide and define the height h of the cylinder, and their horizontal coordinate coincide modulo $\frac{2|\overrightarrow{v}|}{o_t}$, where $o_t = |\Gamma_{up}| \vee |\Gamma_{down}|$, and Γ_{up} (resp. Γ_{down}) is the group of symmetries of the upper (resp. lower) boundary. In general for cylinders appearing in a configuration the orders of these groups are 1 or 2, so o_t is equal to 1 or 2. In the example of the figure above, we have $|\Gamma_{down}| = 2$, $|\Gamma_{up}| = 1$ so $o_t = 2$. So we define the twist as the value $t \in \left[0, \frac{2|\overrightarrow{v}|}{o_t}\right)$ equal to the horizontal coordinates reduced modulo $\frac{2|\overrightarrow{v}|}{o_t}$. This definition will be interesting in the case of general cylinders, that is, cylinders which do not appear necessarily in a configuration, that are used to compute volumes of strata (cf § 4.3).

We have

$$\mathrm{d}\nu_T = \mathrm{d}hol(\hat{\gamma})\mathrm{d}hol(\hat{\delta}_1)\dots\mathrm{d}hol(\hat{\delta}_q).$$

Denote $n(q)$ the number of the cycles $\gamma, \delta_1, \dots, \delta_q$ in $H_1(S, \Sigma, \mathbb{Z})$ that are not homologous to 0 in $H_1(S, \Sigma, \mathbb{Z})$. Taking care of the normalization (Convention 2) we get:

$$\mathrm{d}\nu_T = M_c \cdot \mathrm{d}\overrightarrow{v}\mathrm{d}h_1 \dots \mathrm{d}h_q \mathrm{d}t_1 \dots \mathrm{d}t_q \tag{3.7}$$

with $M_c = 4^{n(q)}$.

Note that with our choice of the basis, $\delta_1, \dots, \delta_q$ are always non homologous to zero. And γ is homologous to zero if and only if the associated graph of the configuration is of type a in the classification of Masur and Zorich (Figure 3 in [MZ]): in this case a vertex corresponding to a cylinder is separating the graph, and the boundary of any cylinder in the configuration consists of a single saddle connection (homologous to γ). So we have:

$$M_c = \begin{cases} 4^q \text{ if } \mathcal{C} \text{ is of type } a \\ 4^{q+1} \text{ otherwise} \end{cases} \tag{3.8}$$

We choose to enumerate the cylinders such that the q_1 first cylinders have a waist curve of holonomy $\pm\overrightarrow{v}$ and the q_2 remaining cylinders have a waist curve of holonomy $\pm 2\overrightarrow{v}$.

Consider now $\mathcal{T}_1^\varepsilon$ the space of parameters of the cylinders with the additional constraint that the sum of the area of the $\hat{\text{h}}$omologous cylinders is normalized (i.e. equal to 1/2) and that $|v|$ is bounded by ε. Then the cone $C(\mathcal{T}_1^\varepsilon)$ underneath $\mathcal{T}_1^\varepsilon$ is given by the following equations:

$$|v|\left(\sum_{k=1}^{q_1} h_k + 2\sum_{k=1}^{q_2} h_{q_1+k}\right) \le \frac{1}{2} \tag{3.9}$$

$$|v| \le \varepsilon\sqrt{2|v|\left(\sum_{k=1}^{q_1} h_k + 2\sum_{k=1}^{q_2} h_{q_1+k}\right)} \tag{3.10}$$

Computation of $c(\mathcal{C})$

The volume of $\mathcal{T}_1^\varepsilon$ is given by:

$$\mathrm{Vol}(\mathcal{T}_1^\varepsilon) = \dim_{\mathbb{R}}(T)\nu_T(C(\mathcal{T}_1^\varepsilon)) = 2(q+1)\nu_T(C(\mathcal{T}_1^\varepsilon))$$

with

$$\nu_T(C(\mathcal{T}_1^\varepsilon)) = \int_{C(\mathcal{T}_1^\varepsilon)} \mathrm{d}\nu_T$$

and $\mathrm{d}\nu_T$ given by (3.7). Note that the measure $\mathrm{d}\overrightarrow{v}$ on $D_\varepsilon/\pm$ disintegrates into $w \cdot \mathrm{d}w \cdot \mathrm{d}\theta$ on $[0,\varepsilon]\times[0,\pi]$, and that integrating the measure of the twists $\mathrm{d}t_1 \dots \mathrm{d}t_q$ on $[0,w)^{q_1} \times \prod_{i=q_1+1}^{q}\left[0, \frac{2w}{o_{t_i}}\right)$ gives a factor $\frac{2^{q_2}}{M_t}w^q$, with

$$M_t = \prod_{i=q_1+1}^{q} o_{t_i}, \tag{3.11}$$

so we get:

$$\nu_T(C(\mathcal{T}_1^\varepsilon)) = M_c\pi\frac{2^{q_2}}{M_t}\int_0^{\frac{\varepsilon}{2}} w^{q+1}\mathrm{d}w \int_{\mathbb{R}_+^q} \chi\left\{\frac{w}{2\varepsilon^2} \le h \le \frac{1}{2w}\right\}\mathrm{d}h_1\dots\mathrm{d}h_q.$$

With the following changes of variables $h'_{q_1+k} = 2h_{q_1+k}$ we obtain:

$$\nu_T(C(\mathcal{T}_1^\varepsilon)) = \frac{M_c}{M_t}\pi\int_0^{\frac{\varepsilon}{2}} w^{q+1}\mathrm{d}w \int_{\mathbb{R}_+^q} \chi\left\{\frac{w}{2\varepsilon^2} \le h' \le \frac{1}{2w}\right\}\mathrm{d}h_1\dots\mathrm{d}h'_q.$$

with

$$h' = \sum_{i=1}^{q_1} h_i + \sum_{i=1}^{q_2} h'_{q_1+i}.$$

Using the fact that

$$\int_{\mathbb{R}_+^q} \chi\left\{a \le \sum_{i=1}^{q} h_i \le b\right\}\mathrm{d}h_1\dots\mathrm{d}h_q = \frac{1}{q!}(b^q - a^q),$$

since it is the difference of the volumes under two simplices in $\mathbb{R}^q$, we obtain after computation:

$$\nu_T(C(\mathcal{T}_1^\varepsilon)) = \frac{M_c\pi\varepsilon^2}{M_t 2^{q+1}}\frac{q}{(q+1)!}$$

Thus:

$$\mathrm{Vol}(\mathcal{T}_1^\varepsilon) = \frac{M_c\pi\varepsilon^2}{M_t 2^q (q-1)!}.$$

We assume now that $\mathcal{Q}(\alpha')$ is non empty, that is, the configuration $\mathcal{C}$ is not made only by cylinders. Let $S' \in \mathcal{Q}_1(\alpha')$, then the rescaled surface $r_S S'$ where $0 < r_S \leq 1$ has area $\frac{r_S^2}{2}$. We define $\Omega(\varepsilon, r_S)$ to be the subset of $\mathcal{T}$ formed by the cylinders rescaled such that gluing them to $r_S S'$ after performing the appropriate surgeries gives a surface $S \in C(\mathcal{Q}_1^\varepsilon(\mathcal{C}))$. Note that the possible variations of area arising when performing the surgeries on $r_S S'$ are negligible ([EMZ] and [MZ]).

By definition $\Omega(\varepsilon, r_S)$ is exactly formed by the rescaled surfaces $r_T T$ where $0 < r_T \leq 1$, $r_T^2 + r_S^2 \leq 1$, and $T \in \mathcal{T}_1^{\tilde{\varepsilon}}$, with $\tilde{\varepsilon} = \varepsilon\sqrt{r_S^2 + r_T^2}$. So we have, denoting $Cusp(\varepsilon) = \mathrm{Vol}(\mathcal{T}_1^\varepsilon)$,

$$\begin{aligned}\nu_T(\Omega(\varepsilon, r_S)) &= \int_0^{\sqrt{1-r_S^2}} r_T^{2n_T-1} Cusp\left(\frac{\tilde{\varepsilon}}{r_T}\right) \mathrm{d}r_T \\ &= \frac{M_c \pi}{M_t 2^q (q-1)!} \int_0^{\sqrt{1-r_S^2}} r_T^{2n_T-1} \varepsilon^2 \frac{r_S^2 + r_T^2}{r_T^2} \mathrm{d}r_T\end{aligned}$$

with $n_T = \dim_{\mathbb{C}}(\mathcal{T}) = q+1$, which simplifies:

$$\nu_T(\Omega(\varepsilon, r_S)) = \frac{M_c \pi \varepsilon^2}{M_t 2^q (q-1)!} \int_0^{\sqrt{1-r_S^2}} r_T^{2q-1}(r_S^2 + r_T^2) \mathrm{d}r_T. \tag{3.12}$$

After computation, we obtain:

$$\nu_T(\Omega(\varepsilon, r_S)) = \frac{M_c \pi \varepsilon^2}{M_t 2^{q+1}(q+1)!}(1 - r_S^2)^q (r_S^2 + q).$$

Now if M_s denote the number of ways to obtain a surface $S \in C(\mathcal{Q}_1^\varepsilon(\mathcal{C}))$ by gluing $r_T T \in \Omega(\varepsilon, r_S)$ to $r_S S' \in \mathcal{Q}(\alpha')$ (see (3.18)), the total measure of the cone $C(\mathcal{Q}_1^\varepsilon(\mathcal{C}))$ is:

$$\begin{aligned}\mu(C(\mathcal{Q}_1^\varepsilon(\mathcal{C}))) &= M_s \,\mathrm{Vol}(\mathcal{Q}_1(\alpha')) \int_0^1 r_S^{2n_S-1} \nu_T(\Omega(\varepsilon, r_S)) \mathrm{d}r_S \\ &= \frac{M_s M_c \,\mathrm{Vol}(\mathcal{Q}_1(\alpha'))\pi\varepsilon^2}{M_t 2^{q+1}(q+1)!} \underbrace{\int_0^1 r_S^{2n_S-1}(r_S^2 + q)(1 - r_S^2)^q \mathrm{d}r_S}_{I}\end{aligned} \tag{3.13}$$

An easy recurrence or a change of variables gives the following lemma:

Lemma 4.

$$J(a, q) = \int_0^1 r^{2a+1}(1 - r^2)^q \mathrm{d}r = \frac{1}{2}\frac{q!a!}{(a+q+1)!}$$

We recognize

$$I = J(n_S, q) + qJ(n_S - 1, q).$$

After simplification we get:

$$I = \frac{(q+1)!(n_S - 1)!}{2(n_S + q + 1)!}(n_S + q).$$

So, denoting $M = \dfrac{M_s M_c}{M_t}$ we obtain:

$$\mu(C(\mathcal{Q}_1^\varepsilon(\mathcal{C}))) = M\pi\varepsilon^2 \,\mathrm{Vol}(\mathcal{Q}_1(\alpha'))\frac{(n_S - 1)!(n_S + q)}{2^{q+2}(n_S + q + 1)!}$$

As we have

$$\mathrm{Vol}\, \mathcal{Q}_1^\varepsilon(\mathcal{C}) = \dim_{\mathbb{R}}(\mathcal{Q}(\alpha))\mu(C(\mathcal{Q}_1^\varepsilon(\mathcal{C})))$$

it follows from the definition of the Siegel–Veech constant that:

$$c(\mathcal{C}) = M \dim_{\mathbb{C}}(\mathcal{Q}(\alpha))\frac{(n_S - 1)!(n_S + q)}{2^{q+2}(n_S + q + 1)!}\frac{\mathrm{Vol}\,\mathcal{Q}_1(\alpha')}{\mathrm{Vol}\,\mathcal{Q}_1(\alpha)}.$$

Recall that $\dim_{\mathbb{C}} \mathcal{Q}(\alpha) = \dim_{\mathbb{C}} \mathcal{Q}(\alpha') + \dim_{\mathbb{C}} \mathcal{T} = n_S + q + 1$. We obtain finally the formula (3.1) of Theorem 8.

Computation of $c_{area}(\mathcal{C})$

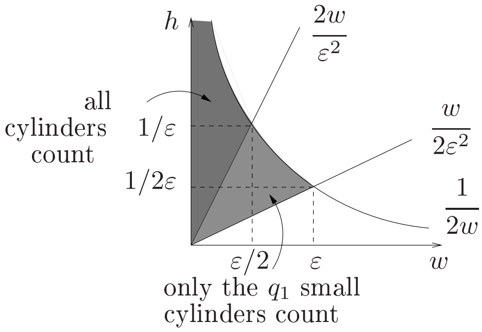
Here we want to compute $c_{area}(\mathcal{C})$, so we have to count surfaces with weight the area of cylinders with waist curve smaller than ε, by definition. Note that, since there are q_1 cylinders of waist curve of length $w = |\overrightarrow{v}|$ and q_2 of waist curve of length $2w$, if $w \leq \frac{\varepsilon}{2}$ (when the area is renormalized), all cylinders count (with weight their area), and if $\frac{\varepsilon}{2} \leq w \leq \varepsilon$, only the thin cylinders count (with weight their area). Equation (3.10) contains two cases

$$w = |v| \leq \frac{\varepsilon}{2}\sqrt{2\text{area}}$$

and

$$\frac{\varepsilon}{2}\sqrt{2\text{area}} \leq w \leq \varepsilon\sqrt{2\text{area}}$$

of different weights. So the domain of integration of $C(T_1^\varepsilon)$ splits into two parts as shown in the following picture.



This gives the following weight function:

$$W^{area}(w, h_i) = \begin{cases} \chi\left\{\dfrac{2w}{\varepsilon^2} \leq h \leq \dfrac{1}{2w}\right\} + \dfrac{\sum_{i=1}^{q_1} h_i}{h}\chi\left\{\dfrac{w}{2\varepsilon^2} \leq h \leq \dfrac{2w}{\varepsilon^2}\right\} & \text{if } w \leq \dfrac{\varepsilon}{2} \\ \dfrac{\sum_{i=1}^{q_1} h_i}{h}\chi\left\{\dfrac{w}{2\varepsilon^2} \leq h \leq \dfrac{1}{2w}\right\} & \text{if } \dfrac{\varepsilon}{2} \leq w \leq \varepsilon \end{cases}$$

with

$$h = \sum_{k=1}^{q_1} h_k + 2\sum_{k=1}^{q_2} h_{q_1+k}.$$

Now the weighted volume of $\mathcal{T}_1^\varepsilon$ is given by:

$$\text{Vol}^{area}(\mathcal{T}_1^\varepsilon) = \dim_{\mathbb{R}}(T)\nu_T^{area}(C(\mathcal{T}_1^\varepsilon)) = 2(q+1)\nu_T^{area}(C(\mathcal{T}_1^\varepsilon))$$

with

$$\nu_T^{area}(C(\mathcal{T}_1^\varepsilon)) = \int_{C(\mathcal{T}_1^\varepsilon)} W^{area}(|\overrightarrow{v}|, h_i)\mathrm{d}\nu_T$$

and $\mathrm{d}\nu_T$ given by (3.7).

Following step by step the computations of the last paragraph, using the same change of variables, we have

$$\begin{aligned}\nu_T^{area}(C(\mathcal{T}_1^\varepsilon)) = \frac{M_c}{M_t}\pi \Bigg[\int_0^{\frac{\varepsilon}{2}} w^{q+1}\mathrm{d}w \int_{\mathbb{R}_+^q} \Bigg(\chi\left\{\frac{2w}{\varepsilon^2} \leq h' \leq \frac{1}{2w}\right\} \\ + \frac{\sum_{i=1}^{q_1} h_i}{h'}\chi\left\{\frac{w}{2\varepsilon^2} \leq h' \leq \frac{2w}{\varepsilon^2}\right\}\Bigg)\mathrm{d}h_1 \ldots \mathrm{d}h'_q \\ + \int_{\frac{\varepsilon}{2}}^{\varepsilon} w^{q+1}\mathrm{d}w \int_{\mathbb{R}_+^q} \frac{\sum_{i=1}^{q_1} h_i}{h'}\chi\left\{\frac{w}{2\varepsilon^2} \leq h' \leq \frac{1}{2w}\right\}\mathrm{d}h_1 \ldots \mathrm{d}h'_q\Bigg].\end{aligned}$$

with

$$h' = \sum_{i=1}^{q_1} h_i + \sum_{i=1}^{q_2} h'_{q_1+i}.$$

Note that, since the variables h_i play symmetric roles, we have:

$$\int_{\mathbb{R}_+^q} \frac{\sum_{i=1}^{q_1} h_i}{\sum_{i=1}^{q} h_i} \chi \left\{ a \leq \sum_{i=1}^{q} h_i \leq b \right\} \mathrm{d}h_1 \ldots \mathrm{d}h_q = \frac{q_1}{q} \int_{\mathbb{R}_+^q} \chi \left\{ a \leq \sum_{i=1}^{q} h_i \leq b \right\} \mathrm{d}h_1 \ldots \mathrm{d}h_q.$$

So computations are similar to the previous ones, and we obtain:

$$\mathrm{Vol}^{area}(\mathcal{T}_1^{\varepsilon}) = \frac{M_c \pi \varepsilon^2}{M_t 2^{q+2} q!}(4q_1 + q_2).$$

Assume that $\mathcal{Q}(\alpha')$ is not empty. Now in (3.12) we have to multiply the integrand by the ratio of the area of the cylinders by the total area of the surface $\frac{r_T^2}{r_S^2 + r_T^2}$. We obtain:

$$\nu_T^{area}(\Omega(\varepsilon, r_S)) = \frac{M_c \pi \varepsilon^2 (4q_1 + q - 2)}{M_t 2^{q+2} q!} \int_0^{\sqrt{1-r_S^2}} r_T^{2q+1} \mathrm{d}r_T = \frac{M_c \pi \varepsilon^2 (4q_1 + q_2)}{M_t 2^{q+2} q!} \frac{(1 - r_S^2)^{q+1}}{2(q+1)}.$$

Then:

$$\mu^{area}(C(Q_1^{\varepsilon}(\mathcal{C}))) = M \,\mathrm{Vol}\, Q_1(\alpha') \frac{\pi \varepsilon^2 (4q_1 + q_2)}{2^{q+3}(q+1)!} \int_0^1 (1 - r_S^2)^{q+1} r_S^{2n_S - 1} \mathrm{d}r_S.$$

Using again Lemma 4 we obtain:

$$\mu^{area}(C(Q_1^{\varepsilon}(\mathcal{C}))) = M \,\mathrm{Vol}\, \mathcal{Q}_1(\alpha') \frac{\pi \varepsilon^2 (4q_1 + q_2)}{2^{q+4}} \frac{(n_S - 1)!}{(n_S + q + 1)!}.$$

So at the end we have:

$$c_{area}(\mathcal{C}) = M \frac{4q_1 + q_2}{2^{q+4}} \frac{(\dim_{\mathbb{C}} \mathcal{Q}(\alpha') - 1)!}{(\dim_{\mathbb{C}} \mathcal{Q}(\alpha) - 1)!} \frac{\mathrm{Vol}\, \mathcal{Q}_1(\alpha')}{\mathrm{Vol}\, \mathcal{Q}_1(\alpha)}. \tag{3.14}$$

Comparing to equation (3.1) and (3.2) we obtain the relation (3.3), which ends the proof of Theorem 8.

Special case

Assume that $\mathcal{Q}(\alpha')$ is empty that is, the configuration is made only by cylinders. This arises only on strata $\mathcal{Q}(-1^4)$, $\mathcal{Q}(2, -1^2)$ and $\mathcal{Q}(2, 2)$. Then the computations are much easier. Indeed we have in this case

$$\mathrm{Vol}\, \mathcal{Q}_1^{\varepsilon}(\mathcal{C}) = \mathrm{Vol}\, \mathcal{T}_1^{\varepsilon} = \frac{M_c \pi \varepsilon^2}{M_t 2^q (q-1)!}$$

and

$$\mathrm{Vol}^{area}\, \mathcal{Q}_1^{\varepsilon}(\mathcal{C}) = \mathrm{Vol}^{area}\, \mathcal{T}_1^{\varepsilon} = \frac{M_c \pi \varepsilon^2}{M_t 2^{q+2} q!}(4q_1 + q_2)$$

so

$$c(\mathcal{C}) = \frac{M_c}{M_t 2^{q+1} (q-1)! \,\mathrm{Vol}\, \mathcal{Q}_1(\alpha)} \tag{3.15}$$

$$c_{area}(\mathcal{C}) = \frac{1}{q} c_{cyl}(\mathcal{C}) = \frac{4q_1 + q_2}{4q} c(\mathcal{C}) \tag{3.16}$$

since the ratio of the area of the cylinders over the total area is 1.

3.3.4 Volume of the boundary strata

Consider a strata $\mathcal{Q}(\alpha) = \prod_{i=1}^m \mathcal{Q}(\alpha_i)$ of disconnected flat surfaces. Following the notations of [AEZ1] and generalizing the result of 4.4 we obtain the following lemma:

Lemma 5.
$$\operatorname{Vol} \mathcal{Q}_1(\alpha) = \frac{1}{2^{m-1}} \frac{\prod(\dim_{\mathbb{C}} \mathcal{Q}(\alpha_i) - 1)!}{(\dim_{\mathbb{C}} \mathcal{Q}(\alpha) - 1)!} \prod_{i=1}^{m} \operatorname{Vol} \mathcal{Q}_1(\alpha_i)$$

We also have the following relation between hyperboloids in the Abelian strata:

Lemma 6.
$$\operatorname{Vol} \mathcal{H}_{1/2}(\alpha) = 2^{\dim_{\mathbb{C}} \mathcal{H}(\alpha)} \operatorname{Vol} \mathcal{H}_1(\alpha)$$

So the final formula for a boundary strata $\mathcal{Q}(\alpha') = \prod \mathcal{H}(\alpha_i) \prod \mathcal{Q}(\beta_j)$ (m connected components) is:

$$c_{area}(\mathcal{C}) = M \frac{4q_1 + q_2}{2^{m+q+3}} \frac{\prod_i (a_i - 1)! 2^{a_i} \operatorname{Vol} \mathcal{H}_1(\alpha_i) \prod_j (b_j - 1)! \operatorname{Vol} \mathcal{Q}_1(\beta_j)}{(\dim_{\mathbb{C}} \mathcal{Q}(\alpha) - 1)! \operatorname{Vol} \mathcal{Q}_1(\alpha)} \tag{3.17}$$

where $a_i = \dim_{\mathbb{C}} \mathcal{H}(\alpha_i)$ and $b_j = \dim_{\mathbb{C}} \mathcal{Q}(\beta_j)$.

3.3.5 Evaluation of M_s

The general formula for M_s is given by:

$$M_s = \frac{K}{|\Gamma(\mathcal{C})|} \tag{3.18}$$

For each surface S_i in the principal boundary, the number of geodesic rays coming from a boundary singularity on S_i can be read on the local ribbon graph representing S_i: each boundary singularity is represented by a connected component of the local ribbon graph, summing the orders k_{i_j} along this connected component gives the number of geodesic rays emerging form this singularity. If the surface as several boundary singularities, then one has to multiply the number of geodesic rays obtained for each of them, to get the combinatorial constant responsible for the gluing of S_i in the configuration. Multiply the numbers obtained for each S_i to get the final combinatorial constant K.

$\Gamma(\mathcal{C})$ denotes the symmetries of the configuration $\mathcal{C}$ generalising the stratum interchange and the cyclic symmetry in the Abelian case. We will explicit this group in the particular cases that we study.

3.4 Strata $\mathcal{Q}(1^k, -1^l)$, with $k - l = 4g - 4 \geq 0$

The strata $\mathcal{Q}(1^k, -1^l)$ are particularly interesting for two reasons. First, they correspond to strata of maximal dimension at genus and number of poles fixed. Second, their boundary strata belong to the same family, so that gives recursion formulae for Siegel–Veech constants and volumes.

The strata $\mathcal{Q}(1^2, -1^2)$ and $\mathcal{Q}(1^4)$ are hyperelliptic and will be studied in § 4.2. In the general case there are only four types of configurations, so we give here their complete description and apply the formula for the Siegel–Veech constant $c_{area}(\mathcal{C})$ to each of them.

3.4.1 Configurations

Proposition 6. *There are only four types of configurations which contain cylinders for strata $\mathcal{Q}(1^k, -1^l)$, they are described in Figure 3.1.*

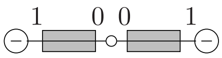
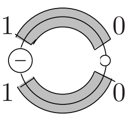
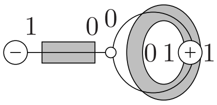
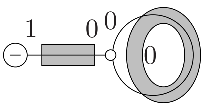
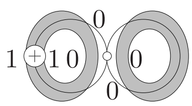
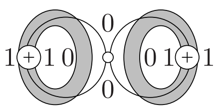
Configurations with cylinders	Boundary strata
General configurations for $g \geq 1$	
$\mathcal{C}_1(k_1,l_1)$ $\{1^{k_1},-1^{l_1-1}\}$  $\{1^{k_2},-1^{l_2-1}\}$ $\begin{cases} l_1+l_2-2=l \\ k_1+k_2+2=k \\ k_1-l_1=4g_1-4 \\ k_2-l_2=4g_2-4 \\ k_i \geq 0,\ l_i \geq 1,\ (k_i,l_i) \neq (1,1) \end{cases}$	$\mathcal{Q}_{g_1}(1^{k_1},-1^{l_1})$ and $\mathcal{Q}_{g_2}(1^{k_2},-1^{l_2})$, for $g_1+g_2=g$
$\mathcal{C}_2$ $\{1^{k-2},-1^{l}\}$ 	$\mathcal{Q}_{g-1}(1^{k-2},-1^{l+2})$, for $k \geq 2$
$\mathcal{C}_3$ $\{1^{k-3},-1^{l}\}$ 	$\mathcal{H}(0)$, $\mathcal{Q}_{g-1}(1^{k-3},-1^{l+1})$ for $k \geq 3$
$\mathcal{C}_4$ $\{1^{k-1},-1^{l-2}\}$ 	$\mathcal{Q}_g(1^{k-1},-1^{l-1})$ for $l \geq 2$
Additional configurations for $g = 1,2$	
$\mathcal{Q}(1^2,-1^2)$ 	$\mathcal{H}(0)$
$\mathcal{Q}(1^4)$ $\emptyset$  $\emptyset$	$\mathcal{H}(0), \mathcal{H}(0)$

Figure 3.1: Configurations containing cylinders for strata $\mathcal{Q}(1^k,-1^l)$, with $k_l = 4g-4$ and $g \geq 1$.

Proof. We recall that graphs representing configurations are classified by Theorem 2 in [MZ]. Then the proof is based on the observation that there not many ways to create zeroes of order 1 or poles (see also Lemma 7 in § 3.5). We recall that the order of a newborn zero is given by the formula $\sum(k_i+1)-2$ where the k_i are the order of the boundary singularities along the boundary component of the ribbon graph that corresponds to the newborn zero (see paragraph 1.4 of [MZ] for more details), and we have $k_i \geq 0$. A boundary component admits at least one boundary singularity. So there is only one possibility for a pole: there is only one boundary singularity, which is equal to 0. For a zero of order 1 there are 3 possibilities:

- One boundary singularity of order 2
- Two boundary singularities of order 1 and 0
- Three boundary singularities of order 0

The first case is realizable when the global graph representing the configuration contain a loop with only one vertex. But in this case we can see that either there will be an other newborn zero of higher order, or there will be no cylinders in the configuration. The third case can be also be eliminated because a boundary components with exactly three boundary singularities arise only around a vertex of type $+3.1$ in the graph, and the parities of the boundary singularities in this case are odd.

So the only remaining possibility is the second one. We can reformulate this discussion by saying that there is only one way to get a cone angle 3π: one has to glue a cone angle π with a cone angle 2π. Looking carefully at all the ways to have boundary singularities of order 1 or 0 in the local ribbon graphs and the consequence on the boundary components in the global graph, we reduce the case to only two possibilities: the boundary singularity of order 0 arises only as cone angle around points on the boundary of a cylinder, and the one of order 1 arises either by creating a hole adjacent to a pole in a surface of non trivial holonomy (i.e. for vertices of type -1.1 and -2.2), or by breaking up a marked point on a surface of trivial holonomy (i.e. for vertices of type $+2.1$). Note that the last surgery creates two points of cone angle π, so gluing each of them to a cylinder will create two newborn zeroes.

This situation is resumed in the following pictures (Figure 3.2).

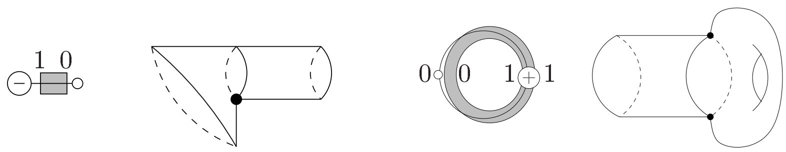


Figure 3.2: Newborn zeroes of order 1

For a pole, similar considerations give that there is only one way to get a pole (and not creating zeroes of order ≥ 2), by pinching the boundary of a cylinder (Figure 3.3).

Figure 3.3: Newborn poles

Note that, since the interior singularities are zeroes of order 1 or poles, the only boundary strata are $\mathcal{H}(0)$ and $\mathcal{Q}(1^K, -1^L)$.

These remarks allow us to eliminate most of the configurations, and to keep only the four possible types of configurations described on Figure 3.1. $\square$

The following tabular details the boundary strata (except $\mathcal{H}(0)$) of a stratum in genus 2.

		Number of poles						
		0	1	2	3	4	5	6
Genus	0	×	×	×	×	$\mathcal{Q}(-1^4)$	$\mathcal{Q}(1,-1^5)$	$\mathcal{Q}(1^2,-1^6)$
	1	×	×	$\mathcal{Q}(1^2,-1^2)$	$\mathcal{Q}(1^3,-1^3)$	$\mathcal{Q}(1^4,-1^4)$	$\mathcal{Q}(1^5,-1^5)$	$\mathcal{Q}(1^6,-1^6)$
	2	$\mathcal{Q}(1^4)$	$\mathcal{Q}(1^5,-1)$	$\mathcal{Q}(1^6,-1^2)$	$\mathcal{Q}(1^7,-1^3)$	$\mathcal{Q}(1^8,-1^4)$	$\mathcal{Q}(1^9,-1^5)$	$\mathcal{Q}(1^{10},-1^6)$

Stratum Boundary strata

In general, the boundary strata of $\mathcal{Q}(1^k,-1^l)$ are those of same genus with at most $l-1$ poles, those of lower genus with at most $l+2$ poles, and $\mathcal{H}(0)$.

Note that, in this list, all values of volumes in genus 0 are known (cf [AEZ1]), and (4.5) gives the values of volumes for the first entries in genus 1 and 2 (hyperelliptic case).

3.4.2 Siegel–Veech constants

Theorem 9. *Let $d=2g-2+k+l=\frac{1}{2}(k+l)$ be the complex dimension of the stratum $\mathcal{Q}(1^k,-1^l)$. The Siegel–Veech constants associated to the four configurations described in Figure 3.1 are the following:*

If $(k_1,l_1)=(k_2,l_2)$:

$$c_{area}(\mathcal{C}_1(k_1,l_1))=\frac{1}{8}\frac{((d_1-1)!)^2}{(d-1)!}\frac{\text{Vol}\,\mathcal{Q}_1(1^{k_1},-1^{l_1})^2}{\text{Vol}\,\mathcal{Q}_1(1^k,-1^l)}$$

Otherwise:

$$c_{area}(\mathcal{C}_1(k_1,l_1))=\frac{1}{4}\frac{(d_1-1)!(d_2-1)!}{(d-1)!}\frac{\text{Vol}\,\mathcal{Q}_1(1^{k_1},-1^{l_1})\,\text{Vol}\,\mathcal{Q}_1(1^{k_2},-1^{l_2})}{\text{Vol}\,\mathcal{Q}_1(1^k,-1^l)}$$

where $d_i=\dim_{\mathbb{C}}\mathcal{Q}(1^{k_i},-1^{l_i})=\frac{1}{2}(3k_i+l_i)$.

$$c_{area}(\mathcal{C}_2)=2\frac{(d-3)!}{(d-1)!}\frac{\text{Vol}\,\mathcal{Q}_1(1^{k-2},-1^{l+2})}{\text{Vol}\,\mathcal{Q}_1(1^k,-1^l)}$$

$$c_{area}(\mathcal{C}_3)=\frac{\pi^2}{3}\frac{(d-5)!}{(d-1)!}\frac{\text{Vol}\,\mathcal{Q}_1(1^{k-3},-1^{l+1})}{\text{Vol}\,\mathcal{Q}_1(1^k,-1^l)}$$

$$c_{area}(\mathcal{C}_4)=\frac{1}{2}\frac{(d-3)!}{(d-1)!}\frac{\text{Vol}\,\mathcal{Q}_1(1^{k-1},-1^{l-1})}{\text{Vol}\,\mathcal{Q}_1(1^k,-1^l)}$$

If all the four configurations appear in a stratum $\mathcal{Q}(1^k,-1^l)$, then the Siegel–Veech constant for the whole stratum is given by:

$$\begin{aligned}
c_{area}(\mathcal{Q}(1^k,-1^l)) \quad &= \sum_{admissible\ (k_1,l_1)} k(k-1)\binom{k-2}{k_1}\binom{l}{l_1-1}c_{area}(\mathcal{C}_1(k_1,l_1))\\
&+ \binom{k}{2}c_{area}(\mathcal{C}_2)+\frac{1}{2}k(k-1)(k-2)c_{area}(\mathcal{C}_3)+\binom{k}{1}\binom{l}{2}c_{area}(\mathcal{C}_4)
\end{aligned}$$

For the additional configurations in genera 1 and 2, see § 4.2.2.

Proof. For each configuration we compute the combinatorial data and apply equation (3.17).

a. Configuration 1 (Figure 3.4):

We have the following combinatorial data for this configuration:

- $M_c=4^1$, $M_t=1$
- $M_s=\frac{1}{|\Gamma|}$ with $|\Gamma|=2$ if $(k_1,l_1)=(k_2,l_2)$, $|\Gamma|=1$ otherwise
- $q_1=1, q_2=0$

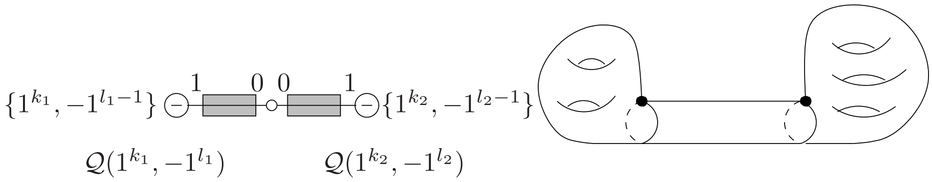


Figure 3.4: Configurations $\mathcal{C}_1(k_1, l_1)$ for $\mathcal{Q}(1^k, -1^l)$ in genus $g \geq 1$

- $\dim_{\mathbb{C}} \mathcal{Q}(1^{k_i}, -1^{l_i}) = 2g_i - 2 + k_i + l_i = \frac{1}{2}(3k_i + l_i)$

Applying Theorem 8 we get:

$$c_{area}(\mathcal{C}_1(k_1, l_1)) = \frac{4}{|\Gamma|}\frac{4}{2^6}\frac{(\frac{3k_1+l_1}{2} - 1)!(\frac{3k_2+l_2}{2} - 1)! \operatorname{Vol} \mathcal{Q}_1(1^{k_1}, -1^{l_1}) \operatorname{Vol} \mathcal{Q}_1(1^{k_2}, -1^{l_2})}{(\frac{3k+l}{2} - 1)! \operatorname{Vol} \mathcal{Q}_1(1^k, -1^l)}$$

Taking care of the numbering of the zeroes, there are $\binom{k}{1} \times \binom{k-1}{1} \times \binom{k-2}{k_1} \times \binom{l}{l_1-1} = k(k-1)\binom{k-2}{k_1} \times \binom{l}{l_1-1}$ configurations $\mathcal{C}_1(k_1, l_1)$.

b. Configuration 2 (Figure 3.5):

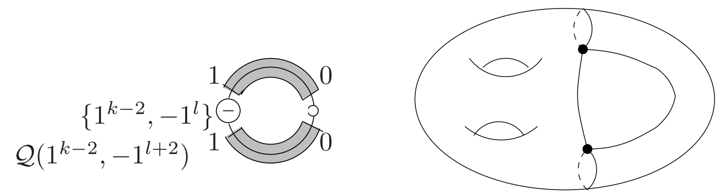


Figure 3.5: Configuration $\mathcal{C}_2$ for $\mathcal{Q}(1^k, -1^l)$ in genus $g \geq 1$ and $k \geq 2$

The combinatorial data are:

- $M_c = 4^2$, $M_t = 1$
- $M_s = 1/|\Gamma| = 1$
- $q_1 = 1, q_2 = 0$
- $\dim_{\mathbb{C}} \mathcal{Q}(1^{k-2}, -1^{l+2}) = 2g + k + l - 4$

We get:

$$c_{area}(\mathcal{C}_2) = 4^2 \frac{4}{2^5} \frac{(2g + k + l - 5)! \operatorname{Vol} \mathcal{Q}_1(1^{k-2}, -1^{l+2})}{(2g + k + l - 3)! \operatorname{Vol} \mathcal{Q}_1(1^k, -1^l)}$$

Taking care of the numbering of the zeroes, there are $\binom{k}{2}$ configurations $\mathcal{C}_2$.

c. Configuration 3 (Figure 3.6):

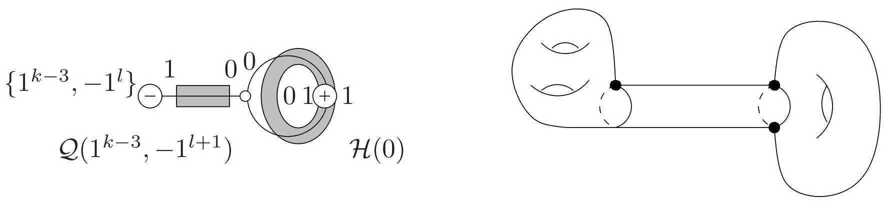


Figure 3.6: Configuration $\mathcal{C}_3$ for $\mathcal{Q}(1^k, -1^l)$ in genus $g \geq 1$ and $k \geq 3$

The combinatorial data are:

- $M_c = 4^2$, $M_t = 1$
- $M_s = 2/|\Gamma| = 1$ because of the involution of $\mathcal{H}(0)$.
- $q_1 = 0, q_2 = 1$
- $\dim_{\mathbb{C}} \mathcal{Q}(1^{k-3}, -1^{l+1}) = 2g + k + l - 6$
- $\mathrm{Vol}\, \mathcal{H}_{1/2}(0) = \frac{4\pi^2}{3}$

We obtain:

$$\begin{aligned} c_{area}(\mathcal{C}_3) &= 4^2 \frac{1}{2^6} \frac{(2g+k+l-7)!\, \mathrm{Vol}\, \mathcal{Q}_1(1^{k-3}, -1^{l+1})(2-1)!\, \mathrm{Vol}\, \mathcal{H}_{1/2}(0)}{(2g+k+l-3)!\, \mathrm{Vol}\, \mathcal{Q}_1(1^k, -1^l)} \\ &= \frac{\pi^2}{3} \frac{(2g+k+l-7)!\, \mathrm{Vol}\, \mathcal{Q}_1(1^{k-2}, -1^{l+2})}{(2g+k+l-3)!\, \mathrm{Vol}\, \mathcal{Q}_1(1^k, -1^l)} \end{aligned}$$

Taking care of the numbering of the zeroes, there are $\binom{k}{1} \times \binom{k-1}{2} = \frac{1}{2}k(k-1)(k-2)$ configurations $\mathcal{C}_3$.

d. Configuration 4 (Figure 3.7):

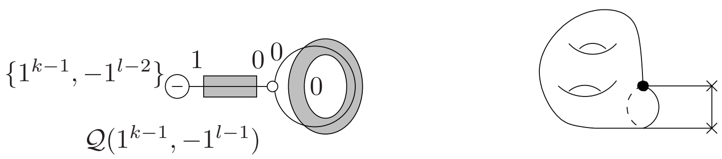


Figure 3.7: Configuration $\mathcal{C}_4$ for $\mathcal{Q}(1^k, -1^l)$ in genus $g \geq 1$ and $l \geq 2$

The combinatorial data are:

- $M_c = 4^2$, $M_t = 1$, $M_s = 1$
- $q_1 = 0, q_2 = 1$
- $\dim_{\mathbb{C}} \mathcal{Q}(1^{k-1}, -1^{l-1}) = 2g + k + l - 4$

Theorem 8 gives:

$$c_{area}(\mathcal{C}_4) = 4^2 \frac{1}{2^5} \frac{(2g+k+l-5)!\, \mathrm{Vol}\, \mathcal{Q}_1(1^{k-1}, -1^{l-1})}{(2g+k+l-3)!\, \mathrm{Vol}\, \mathcal{Q}_1(1^k, -1^l)}$$

Taking care of the numbering of the zeroes, there are $\binom{k}{1} \times \binom{l}{2}$ configurations $\mathcal{C}_4$.

After simplification of the formulae we obtain the results of Theorem 9. □

3.5 Geometry of configurations containing cylinders

This section develops the quadratic version of some geometric results on configurations, proven in the Abelian case in [BG].

3.5.1 Variants of Siegel–Veech constants

The result (3.3) of Theorem 8 can be interpreted as follows: the ratio $\dfrac{c_{area}(\mathcal{C})}{c_{cyl}(\mathcal{C})}$ represents the mean area of a cylinder in configuration $\mathcal{C}$. It does not depend on the configuration, but only on the dimension of the stratum. Summing on all configurations in a stratum we get a result of Vorobets (Theorem 1.6 in [Vo]).

We introduce variants of Siegel–Veech constants whose ratios admit a geometric interpretation. Some of them were introduced by Vorobets.

We define $N_{A_1 \geq p}(S, \mathcal{C}, L)$ (resp. $N_{A \geq p}(S, \mathcal{C}, L)$) that counts configurations $\mathcal{C}$ on S only if the area of a fixed cylinder (resp. all cylinders) filled at least proportion p of the area of the entire surface. As before we denote

$$c_*(\mathcal{C}) = \lim_{L \to \infty} \frac{N_*(S, \mathcal{C}, L) \cdot (\text{Area of } S)}{\pi L^2}$$

the associated Siegel–Veech constants.

We give the analogue of Theorems 4 and 5 of [BG]. Proofs are very similar to the Abelian case so we keep them short.

We introduce the *incomplete Beta function*

$$B(x; n, q) = \int_0^x u^{n-1}(1-u)^{q-1} \mathrm{d}u$$

and the *Beta function* $B(n, q) = B(1; n, q)$. It is a standard fact that

$$B(x; n, q) = B(n, q) \sum_{k=n}^{n+q-1} \binom{n+q-1}{k} x^k (1-x)^{n+q-1-k}.$$

Theorem 10. *Let $\mathcal{C}$ be an admissible configuration for a connected stratum $\mathcal{Q}(\alpha)$ of quadratic differentials. Let q denote the total number of cylinders. Assume that the boundary stratum $\mathcal{Q}(\alpha')$ is non empty, and $q \geq 1$. Then the ratios of Siegel–Veech constants associated to $\mathcal{C}$ are the following:*

$$\frac{c_{A>p}(\mathcal{C})}{c(\mathcal{C})} = \frac{B(1-p; n_S, q)}{B(n_S, q)} \tag{3.19}$$

$$\frac{c_{A_1>p}(\mathcal{C})}{c(\mathcal{C})} = (1-p)^{\dim_{\mathbb{C}} \mathcal{Q}(\alpha) - 2} \tag{3.20}$$

The first ratio can be interpreted as the probability for the cylinders to fill a large part of the area of the surface, and the second ratio the probability for a distinguished cylinder to fill a large part of the area of the surface. Note that the first ratio depends on the number of cylinders q in the configuration, as the second ratio depends only of the dimension of the stratum.

Proof. We begin with the proof of (3.19). We follow step by step the computations of § 3.3.3. The value of $Cusp(\varepsilon)$ does not change. The only adjustment to made is that the area of the surface $r_T T$ which we glue to $r_S S'$ has to satisfy $r_T^2 > p(r_T^2 + r_S^2)$, which is equivalent to $r_T > \sqrt{\frac{p}{1-p}} r_S$. So (3.12) becomes

$$\nu_T^{A>p}(\Omega(\varepsilon, r_S)) = \frac{M_c \pi \varepsilon^2}{M_t 2^q (q-1)!} \int_{\sqrt{\frac{p}{1-p}} r_S}^{\sqrt{1-r_S^2}} r_T^{2q-1}(r_S^2 + r_T^2) \mathrm{d}r_T.$$

and using the constraint $r_T^2 + r_2^2 \leq 1$ we obtain the following bound on r_S: $r_S \leq \sqrt{1-p}$, so (3.13) becomes

$$\mu^{A>p}(C(\mathcal{Q}_1^\varepsilon(\mathcal{C}))) = \frac{M \operatorname{Vol}(\mathcal{Q}_1(\alpha'))\pi\varepsilon^2}{2^{q+1}(q+1)!} \underbrace{\int_0^{\sqrt{1-p}} r_S^{2n_S - 1} \int_{\sqrt{\frac{p}{1-p}} r_S}^{\sqrt{1-r_S^2}} r_T^{2q-1}(r_S^2 + r_T^2) \mathrm{d}r_T \mathrm{d}r_S}_{I_p}$$

Using an appropriate change of variables as the Abelian case, we recognize

$$I_p = \frac{B(1-p; n_S, q)}{4(n_S + q + 1)}$$

where $B(1-p; n_S, q)$ is the incomplete Beta function. Comparing the result to (3.1) we get (3.19).

Now we compute $c_{A_1>p}(\mathcal{C})$: we have the same constraints as before, plus the additional constraint that the first cylinder fills at least part p of the area of the surface. This affects the calculus of the cusp. Note that a cylinder in $S \in \mathcal{Q}_1(\alpha)$ fills at least part p of the surface if it fills at least part $a = p \cdot \frac{r_S^2 + r_T^2}{r_T^2}$ in the space of the cylinders $\mathcal{T}_1$. So we have to replace $Cusp(\varepsilon)$ by

$$\begin{aligned} Cusp^{A_1>a}(\varepsilon) &= 2(q+1)\nu_T^{A_1>p}(C(\mathcal{T}_1^\varepsilon)) \\ &= 2(q+1)\frac{M_c}{M_t}\pi \int_0^{\frac{\varepsilon}{2}} w^{q+1}\mathrm{d}w \int_{\mathbb{R}_+^q} \chi\left\{\frac{w}{2\varepsilon^2} \leq h' \leq \frac{1}{2w}\right\}\chi\{h_1 \geq ah'\}\mathrm{d}h_1 \ldots \mathrm{d}h'_q. \end{aligned}$$

Using the change of variables $h'_1 = h_1 - ah$ we get:

$$Cusp^{A_1>a}(\varepsilon) = Cusp(\varepsilon) \cdot (1-a)^{q-1}$$

Note that if we choose another cylinder, the computations are exactly the same, even if it is a thick cylinder. Finally (3.13) becomes

$$\begin{aligned} \mu^{A_1>p}(C(\mathcal{Q}_1^\varepsilon(\mathcal{C}))) = &\frac{M \operatorname{Vol}(\mathcal{Q}_1(\alpha'))\pi\varepsilon^2}{2^{q+1}(q+1)!} \\ &\cdot \underbrace{\int_0^{\sqrt{1-p}} r_S^{2n_S-1} \int_{\sqrt{\frac{p}{1-p}}r_S}^{\sqrt{1-r_S^2}} r_T^{2q-1}(r_S^2+r_T^2)\left(1 - p\cdot\frac{r_S^2+r_T^2}{r_T^2}\right)\mathrm{d}r_T\mathrm{d}r_S}_{I'_p} \end{aligned}$$

and we get

$$I'_p = \frac{(1-p)^{n_S+q-1}}{4(n_S+q+1)} \cdot B(n,q).$$

Comparing the result to (3.1) we get (3.20). ☐

3.5.2 Maximal number of cylinders

Configurations of quadratic differentials in genus 0 are detailed in [AEZ1]. They contain at most one cylinder. The following proposition gives the maximal number of cylinders in a configuration in higher genus.

Proposition 7. *Consider a stratum $Q(\alpha)$ in genus $g \geq 1$, with $\alpha = (4l_1, \ldots, 4l_m, 4k_1+2, \ldots, 4k_n+2, b_1, \ldots, b_p, -1^k)$, and $l_i \geq 0$, $k_i \geq 0$, b_i odd. First assume that $2n + \sum_{i=1}^p b_i - k + 4 \geq 0$, then the maximal number of $\hat{h}$omologous cylinders satisfies:*

$$q_{max}(\alpha) = \lfloor \frac{n}{2} \rfloor + m + \varepsilon_\alpha,$$

where $\varepsilon_\alpha \in \{0,1,2\}$.

Without this assumption, the maximal number of $\hat{h}$omologous cylinders is given by:

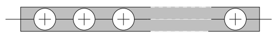
$$\begin{aligned} q_{max}(\alpha) \quad = \quad &\max\{\operatorname{card} I + \frac{\operatorname{card} J}{2};\ I \subset \{1,\ldots,m\}, J \subset \{1,\ldots n\}, \operatorname{card} J \textit{ even}, \\ &4\sum_{i\in I} l_i + 4\sum_{j\in J} k_j + 2n + \sum_{k=1}^p b_k + 4 - k \geq 0\} + \varepsilon_\alpha \end{aligned}$$

To prove this proposition we will need the following lemma:

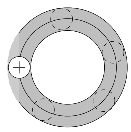
Lemma 7. *Odd zeroes are created by surfaces of non trivial holonomy $\ominus$ or by loops in the graphs of configurations. At most four newborn odd zeroes can be created in a configuration.*

Proof. Since the zeroes on which we perform surgeries on surfaces of non trivial holonomy $\ominus$ are of any order (even or odd), it is easy to see that we can obtain any parity order for newborns zeroes created by surfaces $\ominus$.

This is not the case of surfaces $\oplus$. In fact, a newborn zero represented in the graph by a boundary of a ribbon graph which frames a chain of surfaces $\oplus$ (as in the picture below) surrounded by surfaces $\oplus$ or cylinders has always an even order. This is due to the fact that we perform surgeries such as creating a hole on surfaces of trivial holonomy, so on singularities of cone angle $2k\pi$. If we glue all these surfaces identifying all boundary singularities, then the new cone angle is also multiple of 2π, so the newborn zero is of even order. Boundary types involved in these chains are $\circ 2.2$, $+2.1$, $+2.2$, $+3.2a$, $+3.2b$, $+3.3$, $+4.2a$, $+4.3a$, $+4.4$.



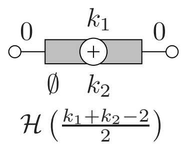
Then we just have to look at the remaining cases, namely, graphs containing surfaces of boundary type $\circ 3.2$ $\circ 4.2$, $+3.1$, $+4.1a$, $+4.1b$, $+4.2b$, $+4.2c$, $+4.3b$. Then one can see case by case that if the ribbon graph is locally as on the picture above, one or two odd zeroes are created (one can replace the surface $\oplus$ by a cylinder $\circ$).

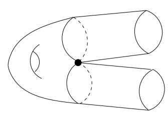


As an example, Figure 3.3 represents how poles are created by loops in the graph of the configuration. $\square$

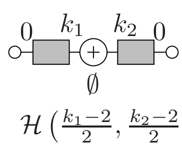
Proof of Proposition 7. This result is a corollary of the classification of configurations of $\hat{\text{h}}$omologous cylinders by Masur and Zorich (Figure 3 in [MZ]). Each configuration is represented by a graph with one, two or three chains of surfaces $\oplus$ (with trivial linear holonomy) and cylinders $\circ$ (see also § 3.2.3 for more details about these graphs). Then there are some remarks:

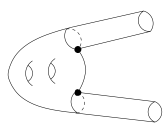
- A surface $\oplus$ of type $+2.1$ (cf Figure 6 in [MZ]) in a chain is surrounded by at most two cylinders. In that case if there is no interior singularity it creates a newborn zero of order $4g = k_1 + k_2 + 2$, where g is the genus of the boundary strata $\mathcal{H}\left(\frac{k_1+k_2-2}{2}\right)$ (k_1 and k_2 are odd).





- A surface $\oplus$ of type $+2.2$ in a chain is surrounded by at most two cylinders and in that case if there is no interior singularity it creates two newborn zeroes of order k_1 and k_2 (even) with $k_1 + k_2 = 4g$ where g is the genus of the boundary strata $\mathcal{H}\left(\frac{k_1-2}{2}, \frac{k_2-2}{2}\right)$





- By Lemma 7, at most 4 zeros of odd order can be realized as newborn zeroes (created by loops in the graph of the configuration or by surfaces $\ominus$), the others are necessarily interior singularities (of surfaces $\ominus$).

- Realizing zeroes as newborn zeroes instead of interior singularities increases the number of cylinders.

First we assume that $2n + \sum_{i=1}^{p} b_i - k + 4 \geq 0$. One procedure to construct the configuration containing the most cylinders is the following: we consider all zeroes of order $4l$ and realize them as newborn zeroes with a surface of type +2.1 as described above. Then we consider the other even zeroes and realize them by pairs as newborn zeroes with surfaces of type +2.2 as described above. At this stage we obtain a chain of $m + \lfloor \frac{n}{2} \rfloor$ surfaces with a cylinder between each surface $\oplus$. We consider the remaining zeroes (at most one even zero and all the odd zeroes). If there are at least 5 odd zeroes, we have to choose graph *a*), *b*) or *c*) following notations of Figure 6 in [MZ] to complete your configuration. If not, we can choose graph *c*), *d*) or *e*). In all cases we will get at most 2 additional cylinders, by looking carefully at all possible configurations depending on the number of odd/even zeroes and poles.

In the general case, we have to choose carefully the even zeroes that we realize as newborn zeroes. Indeed all remaining zeroes should be produced by another surface of non-negative genus in a boundary strata. This condition implies that we can choose to realize zeroes of orders $4l_i$ or pairs of zeroes $4k_{j_1}+2, 4k_{j_2}+2$ with $i \in I$ and $j_1, j_2 \in J$ while $4\sum_{i\in I} l_i + 4\sum_{j\in J} k_j + 2n + \sum_{k=1}^{p} b_k + 4 - k \geq 0$. This explains the general formula for the maximal number of cylinders. $\square$

We are interested in the asymptotic geometry of configurations, in particular when the genus or the number of zeroes tends to infinity, so we will consider $\tilde{q}_{max}(\alpha) = q_{max}(\alpha) - \varepsilon_\alpha$ instead of $q_{max}(\alpha)$, to simplify the computations.

As a corollary of Proposition 7 we obtain that the strata maximizing the number of cylinders at genus fixed are the ones with the most even zeroes:

Corollary 2. *Fix the genus $g \geq 1$ and the number of poles k. Denote $\Pi(4g-4+k)$ the set of partitions α of $4g-4+k$, and $l = \lfloor \frac{k}{4} \rfloor$. Then:*

$$\max_{\alpha \in \Pi(4g-4+k)} \tilde{q}_{max}(\alpha \cup \{-1^k\}) = g + l - 1$$

and the maximum is realized for $\alpha \in \Pi k' \sqcup \Pi_{4,2}(4g-4+4l)$, where $k = 4l + k'$ and $\Pi_{4,2}(4g-4+4l)$ denotes the set of partitions of $4g-4+4l$ using only 4 and 2.

Recall that the mean area of a cylinder is given by $\dfrac{1}{\dim_{\mathbb{C}} \mathcal{Q}(\alpha) - 1}$ (cf Theorem 8), so $\dfrac{q_{max}(\alpha)}{\dim_{\mathbb{C}}(\mathcal{Q}(\alpha)) - 1}$ represents the maximum mean total area of the cylinders in stratum $\mathcal{Q}(\alpha)$. As another corollary of Proposition 7, we obtain Proposition 5.

3.5.3 Configurations with simple surfaces

This section provides an answer in the quadratic case to the following question of Alex Eskin and Alex Wright: for a given stratum or a connected component of a stratum is it possible to find an admissible configuration whose boundary surfaces are only tori ?

Lemma 7 gives the main obstruction to solve this problem in the quadratic case: odd zeroes are created by surfaces of non trivial holonomy $\ominus$ or by loops in graphs of configurations, and there are at most two surfaces of non trivial holonomy or two loops in a configuration. That means that a strata with enough odd zeroes will never have a configuration with only tori as boundary surfaces.

The second obstruction is that, as in the case of abelian differentials, there is no way to have a decomposition into simple surfaces in hyperelliptic components of strata, since they are made from at most two surfaces and cylinders (cf [Bo] and § 4.2).

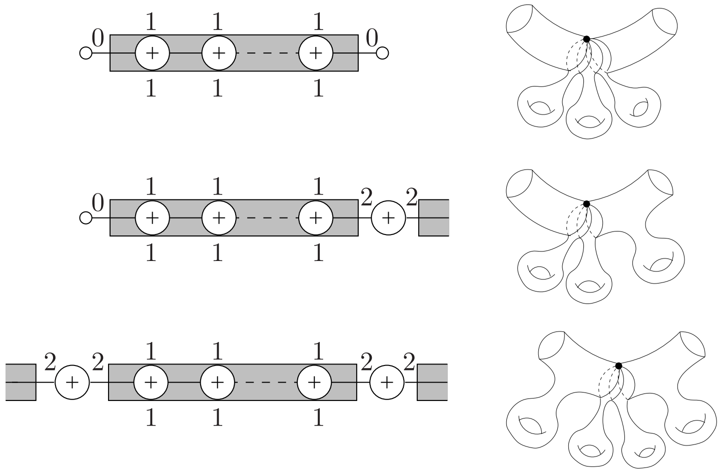
Considering these two obstructions (odd zeroes and hyperelliptic components), we can formulate the following result, which is very similar to the case of Abelian differentials (cf [BG]).

Proposition 8. *Let $\mathcal{Q}^{comp}(\alpha_1, \dots, \alpha_s)$ be a connected component of a stratum of quadratic differentials, which is not hyperelliptic. If all the α_i are even then there exists a configuration in this component containing only tori and cylinders.*

Proof. Denote n the number of zeroes of order $4k+2$ and m the number of zeroes of order $4k$. As in the case of abelian differentials, we just look at what type of zeroes can be created by chains of tori and cylinders. We obtain the same type of zeroes as in the case of Abelian differentials.

For the first type represented in the picture above, the cone angle around the singularity is also $2(2k+1)\pi$ so we obtain a zero of order $4k$.

Zeroes of the second type represented are of order $4k+2$ since the cone angle is $(4k+4)\pi$.



Finally zeroes of the third type are of order $4k+4$.

With these chains we can easily construct a bigger chain that realizes all zeroes. It remains to embed this chain in a graph of configuration. We can see that if there is at least two zeroes of order greater than 4, or if there is at least one zero of order greater than 8, then we can embed this chain in the graph $e)$ with local ribbon graph of type $+4.2a$.

Since $\mathcal{Q}(4)$ is empty, it remains only strata $\mathcal{Q}(2,2,\dots,2)$, which is realizable with a graph of type $e)$ and a local ribbon graph of type $\circ 4.2$, by example.

$\square$

Chapter 4

Volumes of strata or components of strata of quadratic differentials

This chapter corresponds to the second part of a preprint titled
"Siegel–Veech constants and volumes of strata of moduli spaces of quadratic differentials".

This part details certain cases where we can find a closed formula for the volume of a stratum or a connected component of a stratum. In genus 0 the volumes of strata are computed by Athreya, Eskin and Zorich (Theorem 1.6. of [AEZ1]). In higher genus, no closed formula is currently known, even though some calculations are possible using [EOk2] and [EOPa].

The two cases where we can find explicit formulae for the volumes are the hyperelliptic components, using the known volumes in genus 0, and the strata of complex dimension smaller than 5, using techniques developed by Zorich in [Zo3], and by Athreya, Eskin and Zorich in [AEZ2].

Note that in all these cases, we obtain a formula:

$$\mathrm{Vol}\,\mathcal{Q}_1(\alpha) = r \cdot \pi^{2g_{\mathrm{eff}}},\ r \in \mathbb{Q},$$

which is expected to be general. Here $g_{eff} = \hat{g} - g$ where $\hat{g}$ is the genus of the double cover $\hat{S}$.

In these two cases, we apply our previous results and compute Siegel–Veech constants for the entire components or for the entire strata, using values of Siegel–Veech constants for configurations. We choose in purpose examples where we know the values of Siegel–Veech constants in order to show that the results are coherent.

4.1 Volumes of hyperelliptic components

We begin with hyperelliptic components of strata: the values of their volumes are easier to compute since they are related to values of volumes in genus 0, which are computed in [AEZ1].

4.1.1 Volumes of hyperelliptic components of strata of quadratic differentials

The strata of the moduli spaces of quadratic differentials have one or two connected components: for genus $g \geq 5$ there are two components when the stratum contains a hyperelliptic component (cf [La2]). For genus $g \leq 4$ some strata are hyperelliptic and connected (cf [La1]): namely $\mathcal{Q}(1^2, -1^2)$ and $\mathcal{Q}(2, -1^2)$ in genus 1, $\mathcal{Q}(1^4)$, $\mathcal{Q}(2, 1^2)$, and $\mathcal{Q}(2, 2)$ in genus 2. For these strata and for hyperelliptic components of strata in higher genus the volume is easier to compute. We explain here the general strategy to compute these volumes, that we apply in section 4.2.

Proposition 9. *The volumes of hyperelliptic components of strata of quadratic differentials are given by the following formulae:*

- *First type ($k_1 \geq -1$ odd, $k_2 \geq -1$ odd, $(k_1, k_2) \neq (-1,-1)$):*
  If $k_1 \neq k_2$:
$$\mathrm{Vol}^{numb}\,\mathcal{Q}_1^{hyp}(k_1^2, k_2^2) = \frac{2^d}{(d)!}\pi^d \frac{k_1!!}{(k_1+1)!!}\frac{k_2!!}{(k_2+1)!!} \tag{4.1}$$
  Otherwise:
$$\mathrm{Vol}^{numb}\,\mathcal{Q}_1^{hyp}((g-1)^4) = \frac{3\cdot 2^{2g+2}}{(2g+2)!}\pi^{2g+2}\left(\frac{(g-1)!!}{g!!}\right)^2 \tag{4.2}$$
- *Second type ($k_1 \geq -1$ odd, $k_2 \geq 0$ even):*
$$\mathrm{Vol}^{numb}\,\mathcal{Q}_1^{hyp}(k_1^2, 2k_2+2) = \frac{2^d}{(d)!}\pi^{d-1} \frac{k_1!!}{(k_1+1)!!}\frac{k_2!!}{(k_2+1)!!} \tag{4.3}$$
- *Third type (k_1, k_2 even):*
$$\mathrm{Vol}^{numb}\,\mathcal{Q}_1^{hyp}(2k_1+2, 2k_2+2) = \frac{2^{d+1}}{(d)!}\pi^{d-2} \frac{k_1!!}{(k_1+1)!!}\frac{k_2!!}{(k_2+1)!!} \tag{4.4}$$

where $d = k_1 + k_2 + 4$ is the complex dimension of the strata.

Example 1. For the five strata that are connected and hyperelliptic we obtain:

$$\mathrm{Vol}\,\mathcal{Q}_1(1^2, -1^2) = \frac{\pi^4}{3} = 30\zeta(4) \qquad \mathrm{Vol}\,\mathcal{Q}_1(1^4) = \frac{\pi^6}{15} = 63\zeta(6) \tag{4.5}$$
$$\mathrm{Vol}\,\mathcal{Q}_1(2, -1^2) = \frac{4\pi^2}{3} = 8\zeta(2) \quad \mathrm{Vol}\,\mathcal{Q}_1(2, 1^2) = \frac{2\pi^4}{15} = 12\zeta(4) \tag{4.6}$$
$$\mathrm{Vol}\,\mathcal{Q}_1(2, 2) = \frac{4\pi^2}{3} = 8\zeta(2) \tag{4.7}$$

For an alternative computation of some of these volumes using graphs, see section 4.7.

Proof. By Convention 3 we compute volumes of strata with numbered zeroes. We denote $\mathrm{Vol}^{numb}\,\mathcal{Q}(\alpha)$ the volume of the strata $\mathcal{Q}(\alpha)$ when the zeroes are numbered and $\mathrm{Vol}^{unnumb}\,\mathcal{Q}(\alpha)$ when they are not. We have the following relation:

$$\mathrm{Vol}^{numb}\,\mathcal{Q}_1(d_1^{\alpha_1}, d_2^{\alpha_2}, \dots, d_m^{\alpha_m}) = \frac{\alpha_1!\alpha_2!\dots\alpha_m!}{|\Gamma(\alpha)|}\mathrm{Vol}^{unnumb}\,\mathcal{Q}_1(d_1^{\alpha_1}, d_2^{\alpha_2}, \dots, d_m^{\alpha_m})$$

where $\Gamma(\alpha)$ denotes the group of possible symmetries of all surfaces in the stratum $\mathcal{Q}(\alpha)$.

We recall here the three types of strata that contain hyperelliptic components (cf [La1]):

- First type:
$$\mathcal{Q}_g^{hyp}(k_1^2, k_2^2) \xrightarrow{\pi} \mathcal{Q}_0(k_1, k_2, -1^{2g+2})$$
  for $k_1 \geq -1$ odd, $k_2 \geq -1$ odd, $(k_1, k_2) \neq (-1,-1)$, $g = \frac{1}{2}(k_1+k_2)+1$. The ramification points are $2g+2$ poles. Note that for $k_i = -1$ there are $2g+3$ poles and $\binom{2g+3}{1}$ choices for the cover, so in that case π is $(2g+3):1$.
- Second type:
$$\mathcal{Q}_g^{hyp}(k_1^2, 2k_2+2) \xrightarrow{\pi} \mathcal{Q}_0(k_1, k_2, -1^{2g+1})$$
  for $k_1 \geq -1$ odd, $k_2 \geq 0$ even, $g = \frac{1}{2}(k_1+k_2+3)$. The ramification points are $2g+1$ poles and the zero of order k_2. Note that for $k_1 = -1$ there are $2g+2$ poles and $\binom{2g+2}{1}$ choices for the cover, so in that case π is $(2g+2):1$.
- Third type:
$$\mathcal{Q}_g^{hyp}(2k_1+2, 2k_2+2) \xrightarrow{\pi} \mathcal{Q}_0(k_1, k_2, -1^{2g})$$
  for k_1, k_2 even, $g = \frac{1}{2}(k_1+k_2)+2$. The ramification points are over all the singularities.

Except the special cases, π is always $1:1$.

We introduce the following notation for the general case:

$$\mathcal{Q}^{hyp}(\alpha) \xrightarrow[I:1]{\pi} \mathcal{Q}(\beta)$$

with $\alpha = (d_1^{\alpha_1}, \dots, d_n^{\alpha_n})$ and $\beta = (\tilde{d}_1^{\beta_1}, \dots, \tilde{d}_m^{\beta_m})$.

Let $d = \dim_{\mathbb{C}} \mathcal{Q}(\beta)$ be the complex dimension of the stratum that we consider.

Recall that, by definition, the volume of the hyperboloid of surfaces of area equal to $1/2$ is given by the volume of the cone underneath times the real dimension of the stratum:

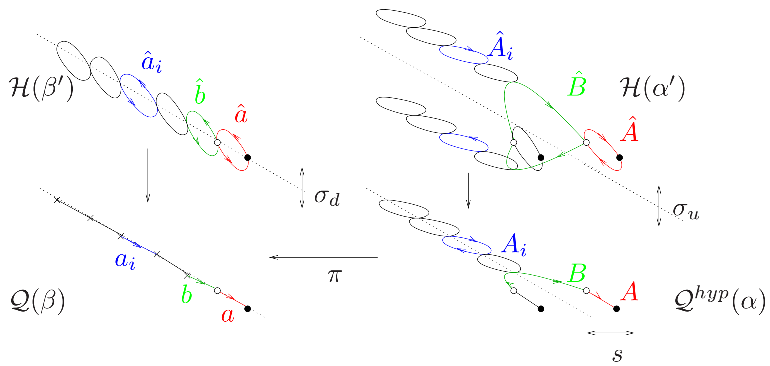
$$\operatorname{Vol} \mathcal{Q}_1(\beta) = 2d \cdot \operatorname{Vol}\{S \in \mathcal{Q}(\beta),\ \operatorname{area}(S) \leq 1/2\}$$

Let S be a point in $\mathcal{Q}_1(\beta)$, and let S' be one of the I possible lifts $\pi^*(S)$. As S is of area $1/2$, S' is of area 1 so belongs to $\mathcal{Q}_2^{hyp}(\alpha)$. So the cone underneath $\mathcal{Q}_1(\beta)$ is in $1:I$ correspondence with the cone underneath $\mathcal{Q}_2^{hyp}(\alpha)$. Now we want to compare the volume elements of $\mathcal{Q}^{hyp}(\alpha)$ and $\mathcal{Q}(\beta)$. So we have to understand how the lattice $(H_1^-(\hat{S}, \hat{\Sigma}; \mathbb{Z}))^*_{\mathbb{C}}$ is lifted by π and compare it with the lattice $(H_1^-(\hat{S}', \hat{\Sigma}'; \mathbb{Z}))^*_{\mathbb{C}}$.

For the first type we have the following commutative diagram:

$$\begin{tikzcd}
\mathcal{H}_{g+1}(k_1+1, k_2+1) \arrow[d] & \mathcal{H}_{2g+1}((k_1+1)^2, (k_2+1)^2) \arrow[l, dotted] \arrow[d] \\
\mathcal{Q}_0(k_1, k_2, -1^{2g+2}) & \mathcal{Q}_g^{hyp}(k_1^2, k_2^2) \arrow[l, "\pi"', "I:1"]
\end{tikzcd}$$

On $S \in \mathcal{Q}_0(k_1, k_2, -1^{2g+2})$ we consider the saddle connections defined by taking a broken line joining all the singularities except one pole, as in the picture above, such that a joins the two zeroes, b joins a zero to a pole, and a_i, b_i join the remaining poles except the last one, for i going from 1 to g. Then $\hat{a}, \hat{b}, \hat{a}_1, \dots, \hat{b}_g$ is a primitive basis of $H_1^-(\hat{S}, \hat{\Sigma}; \mathbb{Z})$ (cf Lemma 3). On the other hand consider the saddle connections on $\mathcal{Q}_g^{hyp}(k_1^2, k_2^2)$ constructed using $a, b, a_1, \dots b_g$ in the following way: for all a_i and b_i and for b, take the combination of the two lifts by π to obtain primitive cycles A_i, B_i, and B in $H_1(S', \Sigma', \mathbb{Z})$. Take only one of the two preimages of a to get a primitive cycle A. Then $\hat{A}, \hat{B}, \hat{A}_1, \dots, \hat{B}_g$ define a primitive basis of $H_1^-(\hat{S}', \hat{\Sigma}'; \mathbb{Z})$ (same arguments as in Lemma 3).



On the picture σ_u and σ_d are the involutions of the double covers and s is the hyperelliptic involution.

In this local coordinates volume elements are given by:

$$\mathrm{d}\nu_{down} = \mathrm{d}\hat{a}\,\mathrm{d}\hat{b}\,\mathrm{d}\hat{a}_1 \dots \mathrm{d}\hat{b}_g = 2^{2d}\mathrm{d}a\,\mathrm{d}b\,\mathrm{d}a_1 \dots \mathrm{d}b_g$$

and

$$\mathrm{d}\nu_{up} = \mathrm{d}\hat{A}\,\mathrm{d}\hat{B}\,\mathrm{d}\hat{A}_1 \dots \mathrm{d}\hat{B}_g = 2^{2d}\mathrm{d}A\,\mathrm{d}B\,\mathrm{d}A_1 \dots \mathrm{d}B_g$$

with $\mathrm{d}A = \pi^*(\mathrm{d}a)$, $\mathrm{d}B = 4\pi^*(\mathrm{d}b)$, $\mathrm{d}A_i = 4\pi^*(\mathrm{d}a_i)$ and $\mathrm{d}B_i = 4\pi^*(\mathrm{d}b_i)$.

So we obtain the following relation between the volume elements:

$$\mathrm{d}\nu_{up} = 2^{2d-2}\pi^*(\mathrm{d}\nu_{down})$$

Same considerations for the two other types give the same result.
So now we have all the elements to compute the relation between $\mathrm{Vol}^{numb}\,\mathcal{Q}_1(\beta)$ and $\mathrm{Vol}^{numb}\,\mathcal{Q}_1^{hyp}(\alpha)$:

$$\begin{aligned}
\mathrm{Vol}^{numb}\,\mathcal{Q}_1^{hyp}(\alpha) &= \frac{\alpha_1!\dots\alpha_n!}{|\Gamma^{hyp}(\alpha)|}\mathrm{Vol}^{unnumb}\,\mathcal{Q}_1^{hyp}(\alpha)\\
&= \frac{\alpha_1!\dots\alpha_n!}{|\Gamma^{hyp}(\alpha)|}\cdot 2d\,\mathrm{Vol}^{unnumb}\{S'\in\mathcal{Q}^{hyp}(\alpha),\ \mathrm{area}(S')\le 1/2\}\\
&= \frac{\alpha_1!\dots\alpha_n!}{|\Gamma^{hyp}(\alpha)|}\cdot 2d\cdot\frac{1}{2^d}\mathrm{Vol}^{unnumb}\{S\in\mathcal{Q}^{hyp}(\alpha),\ \mathrm{area}(S)\le 1\}\\
&= \frac{\alpha_1!\dots\alpha_n!}{|\Gamma^{hyp}(\alpha)|}\cdot 2d\cdot\frac{1}{2^d}\cdot I\cdot 2^{2d-2}\,\mathrm{Vol}^{unnumb}\{S\in\mathcal{Q}(\beta),\ \mathrm{area}(S)\le 1/2\}\\
&= \frac{\alpha_1!\dots\alpha_n!}{|\Gamma^{hyp}(\alpha)|}\cdot I\cdot 2^{d-2}\,\mathrm{Vol}^{unnumb}\,\mathcal{Q}_1(\beta)\\
&= \frac{\alpha_1!\dots\alpha_n!}{|\Gamma^{hyp}(\alpha)|}\cdot I\cdot 2^{d-2}\cdot\frac{|\Gamma(\beta)|}{\beta_1!\dots\beta_m!}\mathrm{Vol}^{numb}\,\mathcal{Q}_1(\beta)
\end{aligned}$$

Note that, for the first two types, the hyperelliptic involution exchanges the zeroes which are preimages of the same zero downstairs. So for these types $|\Gamma^{hyp}(\alpha)| = 2$. For the third type $|\Gamma^{hyp}(\alpha)| = 1$. Downstairs there is no symmetry for each stratum that we consider so $|\Gamma(\beta)| = 1$ for each β.

The values of the volumes of strata of quadratic differentials in genus 0 are given in [AEZ1], Theorem 1.6:

$$\mathrm{Vol}\,\mathcal{Q}_1(d_1,\dots,d_n) = 2\pi^2\prod_{i=1}^{n} v(d_i), \tag{4.8}$$

with

$$v(n) = \frac{n!!}{(n+1)!!}\cdot\pi^n\cdot\begin{cases}\pi & \text{when } n \text{ is odd}\\ 2 & \text{when } n \text{ is even}\end{cases}$$

for $n\in\{-1,0\}\cup\mathbb{N}$ and with

$$n!! = n(n-2)(n-4)\cdots,$$

by convention $(-1)!! = 0!! = 1$.

In particular we have:

- for the first type ($k_1\ge -1$ odd, $k_2\ge -1$ odd, $(k_1,k_2)\ne(-1,-1)$, $d = 2g+2$):

$$\mathrm{Vol}^{numb}\,\mathcal{Q}_1(k_1,k_2,-1^d) = 2\pi^d\frac{k_1!!}{(k_1+1)!!}\cdot\frac{k_2!!}{(k_2+1)!!},$$

- for the second type ($k_1\ge -1$ odd, $k_2\ge 0$ even, $d = 2g+1$):

$$\mathrm{Vol}^{numb}\,\mathcal{Q}_1(k_1,k_2,-1^d) = 4\pi^{d-1}\frac{k_1!!}{(k_1+1)!!}\cdot\frac{k_2!!}{(k_2+1)!!},$$

- for the third type (k_1, k_2 even, $d = 2g$):

$$\mathrm{Vol}^{numb}\,\mathcal{Q}_1(k_1,k_2,-1^d) = 8\pi^{d-2}\frac{k_1!!}{(k_1+1)!!}\cdot\frac{k_2!!}{(k_2+1)!!}.$$

So we obtain the result. □

4.1.2 Volumes of hyperelliptic components of strata of Abelian differentials

Proposition 10. *The volumes of hyperelliptic components of strata of Abelian differentials with area 1/2 are given by the following formulae:*

$$\mathrm{Vol}^{numb}\,\mathcal{H}_{1/2}^{hyp}(k-1) = \frac{2^{k+2}}{(k+2)!}\cdot\frac{(k-2)!!}{(k-1)!!}\cdot\pi^{k+1} \tag{4.9}$$

$$\mathrm{Vol}^{numb}\,\mathcal{H}_{1/2}^{hyp}\left(\left(\frac{k}{2}-1\right)^2\right) = \frac{2^{k+3}}{(k+2)!}\cdot\frac{(k-2)!!}{(k-1)!!}\cdot\pi^{k} \tag{4.10}$$

Proof. We recall here the two types of strata of Abelian differentials that contain hyperelliptic components (cf [KZ]):

- First type ($g \geq 2$):
$$\mathcal{H}^{hyp}(2g-2) \xrightarrow{\pi} \mathcal{Q}(2g-3,-1^{2g+1})$$
- Second type ($g \geq 2$):
$$\mathcal{H}^{hyp}((g-1)^2) \xrightarrow{\pi} \mathcal{Q}(2g-2,-1^{2g+2})$$

In both cases, π is an isomorphism. By conventions 2 and 1, the volume elements are chosen to be invariant under this isomorphism, so we have:

$$\mathrm{Vol}^{unnumb}\,\mathcal{H}_1^{hyp}(2g-2) = \mathrm{Vol}^{unnumb}\,\mathcal{Q}_1(2g-3,-1^{2g+1})$$
$$\mathrm{Vol}^{unumb}\,\mathcal{H}_1^{hyp}((g-1)^2) = \mathrm{Vol}^{unnumb}\,\mathcal{Q}_1(2g-2,-1^{2g+2})$$

So considering the naming of the singularities we obtain:

$$\begin{aligned}
\mathrm{Vol}^{numb}\,\mathcal{H}_1^{hyp}(2g-2) &= \frac{1}{(2g+1)!}\mathrm{Vol}^{numb}\,\mathcal{Q}_1(2g-3,-1^{2g+1})\\
&= \frac{2}{(2g+1)!}\cdot\frac{(2g-3)!!}{(2g-2)!!}\cdot\pi^{2g}\\
\mathrm{Vol}^{numb}\,\mathcal{H}_1^{hyp}((g-1)^2) &= \frac{2!}{2}\mathrm{Vol}^{unnumb}\,\mathcal{H}_1^{hyp}((g-1)^2)\\
&= \frac{2}{(2g+2)!}\mathrm{Vol}^{numb}\,\mathcal{Q}_1(2g-2,-1^{2g+2})\\
&= \frac{8}{(2g+2)!}\cdot\frac{(2g-2)!!}{(2g-1)!!}\cdot\pi^{2g}
\end{aligned}$$

by plugging values of volumes given in (4.8). For the first type, for $k = 2g-1$ we have $\dim_{\mathbb{C}}\mathcal{H}(k-1) = 2g = k+1$. For the second type, for $k = 2g$ we have $\dim_{\mathbb{C}}\mathcal{H}\left(\left(\frac{k}{2}-1\right)^2\right) = 2g+1 = k+1$. Finally, using Lemma 6 we obtain the result. □

4.2 Coherence of the formulae for the hyperelliptic components of strata

4.2.1 Configurations containing cylinders in hyperelliptic components

The complete list of all configurations of ĥomologous saddle connexions is described by C. Boissy in [Bo]. We extract from this list the configurations containing cylinders, and recall them on Figure 4.1.

The following proposition precise the boundary of the hyperelliptic components of strata.

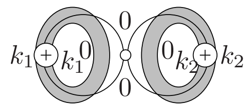
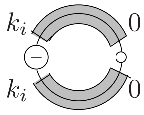
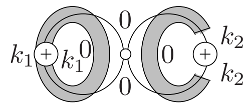
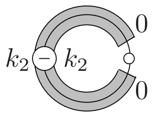
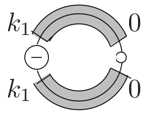
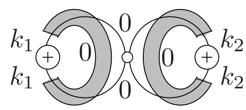
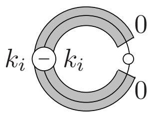
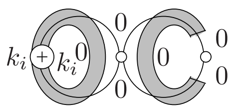
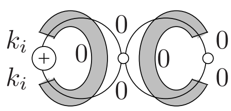
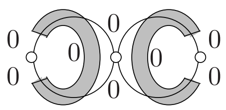
Configurations with cylinders	Boundary strata
$\mathcal{Q}^{hyp}(k_1^2, k_2^2)$, k_1, k_2 odd, $(k_1, k_2) \neq (-1, -1)$	
$\emptyset$  $\emptyset$ $\mathcal{C}_1$	$\mathcal{H}^{hyp}(k_1 - 1)$, $\mathcal{H}^{hyp}(k_2 - 1)$
$\{k_j^2\}$  $\mathcal{C}_2(k_i)$	$\mathcal{Q}^{hyp}_{g-1}(k_j^2, (k_i - 2)^2)$
$\mathcal{Q}^{hyp}(k_1^2, 2k_2 + 2)$, k_1 odd and k_2 even	
$\emptyset$  $\emptyset$ $\mathcal{C}_1$	$\mathcal{H}^{hyp}(k_1 - 1)$, $\mathcal{H}^{hyp}((\frac{k_2}{2} - 1)^2)$
$\{k_1^2\}$  $\mathcal{C}_2$	$\mathcal{Q}^{hyp}_{g-1}(k_1^2, 2k_2 - 2)$
$\{2k_2 + 2\}$  $\mathcal{C}_3$	$\mathcal{Q}^{hyp}_{g-1}(2k_2 + 2, (k_1 - 2)^2)$
$\mathcal{Q}^{hyp}(2k_1 + 2, 2k_2 + 2)$, k_1, k_2 even	
$\emptyset$  $\emptyset$ $\mathcal{C}_1$	$\mathcal{H}^{hyp}((\frac{k_1}{2} - 1)^2)$, $\mathcal{H}^{hyp}((\frac{k_2}{2} - 1)^2)$
$\{2k_j + 2\}$  $\mathcal{C}_2(k_i)$	$\mathcal{Q}^{hyp}_{g-1}(2k_j + 2, 2k_i - 2)$
Additional configurations	
$\emptyset$  $\emptyset$ $\mathcal{Q}^{hyp}(k_i^2, 2)$	$\mathcal{H}^{hyp}(k_i - 1)$
$\emptyset$  $\mathcal{Q}^{hyp}(2k_i + 2, 2)$	$\mathcal{H}^{hyp}((\frac{k_i}{2} - 1)^2)$
 $\mathcal{Q}(2, 2)$	$\emptyset$

Figure 4.1: Configurations containing cylinders for hyperelliptic components of strata of quadratic differentials

Proposition 11. *Let S be a flat surface in a hyperelliptic component of a stratum of quadratic differentials $\mathcal{Q}^{hyp}(\alpha)$. Les $\hat{\gamma}$ be a collection of homologous saddle connexions realizing a configuration $\mathcal{C}$ on the previous list (Figure 4.1). Then the two possible boundary components $S_1, S_2 \in \mathcal{Q}(\alpha'_1), \mathcal{Q}(\alpha'_2)$ of S are hyperelliptic.*

For every surfaces $S_1 \in \mathcal{Q}^{hyp}(\alpha'_1)$, $S_2 \in \mathcal{Q}^{hyp}(\alpha'_2)$, there is at least one way to assemble S_1 and eventually S_2 following configuration $\mathcal{C}$ to obtain a hyperelliptic surface S.

Proof. If $S \in \mathcal{Q}^{hyp}(\alpha)$, following Lemma 10.3 of [EMZ], we may assume that the hyperelliptic involution fixes each boundary component. So it implies that S_1 and S_2 are also hyperelliptic.

If $S_1 \in \mathcal{Q}^{hyp}(\alpha'_1)$ and $S_2 \in \mathcal{Q}^{hyp}(\alpha'_2)$, we can make the surgeries on the boundary surfaces in such a way that the new surfaces stay invariant under the hyperelliptic involution (cf § 14 in [EMZ]). Then we construct an application on S that acts on each boundary component as the hyperelliptic involution for the corresponding stratum and on the cylinder either by fixing its boundaries and rotating or by exchanging its two boundaries depending on the configuration $\mathcal{C}$, in such a way that the global application is an involution of S. The action of the hyperelliptic involution on the configurations is detailed in [Bo]. □

4.2.2 Siegel–Veech constants for configurations in hyperelliptic components

Note that the complex dimension of any hyperelliptic component is given by: $\dim_{\mathbb{C}} \mathcal{Q}^{hyp}(k_1^2, 2k_2+2) = \dim_{\mathbb{C}} \mathcal{Q}^{hyp}(k_1^2, k_2^2) = \dim_{\mathbb{C}} \mathcal{Q}^{hyp}(2k_1+2, 2k_2+2) = k_1+k_2+4 =: d$.

First recall that the constants for the entire components are known ([EKZ2]):

Lemma 8.
$$c_{area}(\mathcal{Q}^{hyp}(\alpha)) = \frac{k_1+k_2+4}{4\pi^2}\left(2+\frac{1}{(k_1+2)(k_2+2)}\right) \tag{4.11}$$
for $\alpha = (k_1^2, k_2^2)$, $\alpha = (k_1^2, 2k_2+2)$ or $\alpha = (2k_1+2, 2k_2+2)$.

Proof. It is a direct consequence of Corollary 3 in [EKZ2]. We L^- denote the sum of the Lyapunov exponents $\lambda_1^-, \ldots, \lambda_{g_{\text{eff}}}^-$ for the hyperelliptic component $\mathcal{Q}^{hyp}(\alpha)$. Recall that by Theorem 1 of [EKZ2], we have:
$$c_{area}(\mathcal{Q}^{hyp}(\alpha)) = \frac{3}{\pi^2}(L^- - I - K)$$
where
$$I = \frac{1}{4}\sum_{d_j\ odd}\frac{1}{d_j+2}, \quad K = \frac{1}{24}\sum_j \frac{d_j(d_j+4)}{d_j+2}.$$

Corollary 3 in [EKZ2] gives the values of L^- for hyperelliptic components, that we recall here:
$$L^- = \frac{k_1+k_2+4}{4}\left(1+\frac{1}{(k_1+2)(k_2+2)}\right) \quad \text{for} \quad \mathcal{Q}^{hyp}(k_1^2, k_2^2)$$
$$L^- = \frac{k_1+k_2+4}{4}\left(1+\frac{1}{k_1+2}\right) \quad \text{for} \quad \mathcal{Q}^{hyp}(k_1^2, 2k_2+2)$$
$$L^- = \frac{k_1+k_2+4}{4} \quad \text{for} \quad \mathcal{Q}^{hyp}(2k_1+2, 2k_2+2)$$
□

Now to compute these constants for each configuration, we use the method described in § 3.2.5, we follow step by step the computations of § 3.3 and make only a few adjustments.

First the boundary of $\mathcal{Q}^{hyp}(\alpha)$ is described by Proposition 11 and consists of hyperelliptic components of the boundary strata of $\mathcal{Q}(\alpha)$, so $\text{Vol}_* \mathcal{Q}_1^{\varepsilon}(comp, \mathcal{C})$ will be express in terms of $\prod_i \text{Vol}\, \mathcal{Q}^{hyp}(\alpha'_i)$. Second we will have to take care of the symmetries induced by the hyperelliptic involution.

Finally we obtain the following variation of formula (3.17):

$$c_{area}(\mathcal{C}) = M\frac{4q_1+q_2}{2^{m+q+3}}\frac{\prod_i (a_i-1)!2^{a_i}\operatorname{Vol}\mathcal{H}_1^{hyp}(\alpha_i)\prod_j (b_j-1)!\operatorname{Vol}\mathcal{Q}_1^{hyp}(\beta_j)}{(\dim_{\mathbb{C}}\mathcal{Q}(\alpha)-1)!\operatorname{Vol}\mathcal{Q}_1^{hyp}(\alpha)} \tag{4.12}$$

where $M = \frac{M_s M_c}{M_t}$ and M_c, M_t are given by (3.8) and (3.11), and M_s takes care of the hyperelliptic involution. It will be detailed for each configuration in the following paragraphs.

$\mathcal{Q}^{hyp}(k_1^2, k_2^2)$, with k_1 and k_2 odd and $(k_1,k_2) \neq (-1,-1)$

Proposition 12. *For the first type of hyperelliptic components, the Siegel–Veech constants for the configurations described on Figure 4.1 are given by the following formulae:*

$$c_{area}(\mathcal{C}_1) = \frac{k_1+k_2+4}{4(k_1+2)(k_2+2)\pi^2} \tag{4.13}$$

$$c_{area}(\mathcal{C}_2(k_i)) = \frac{(k_1+k_2+4)(k_i+1)}{(k_1+k_2+2)2\pi^2} \tag{4.14}$$

Moreover:

$$c_{area}(\mathcal{Q}^{hyp}(k_1^2,k_2^2)) = c_{area}(\mathcal{C}_1) + c_{area}(\mathcal{C}_2(k_1)) + c_{area}(\mathcal{C}_2(k_2))$$

Example 2.

$$c_{area}(\mathcal{Q}(1^2,(-1)^2)) = c_{area}(\mathcal{C}_1) + c_{area}(\mathcal{C}_2(1)) = \frac{1}{3\pi^2} + \frac{2}{\pi^2} = \frac{7}{3\pi^2}$$

$$c_{area}(\mathcal{Q}(1^4)) = c_{area}(\mathcal{C}_1) + 2c_{area}(\mathcal{C}_2(1)) = \frac{1}{6\pi^2} + \frac{3}{\pi^2} = \frac{19}{6\pi^2}$$

Proof. For each configuration on the list (Figure 4.1) we apply the formula (4.12).

a. Configuration $\mathcal{C}_1$ for $k_1 \geq 1, k_2 \geq 1$:

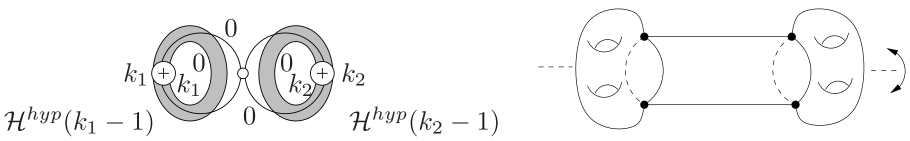


Figure 4.2: Configuration $\mathcal{C}_1$ for $\mathcal{Q}^{hyp}(k_1^2,k_2^2)$

- $M_c = 4^2$, $M_t = 1$
- $M_s = \dfrac{4k_1k_2}{|\Gamma|}$, $|\Gamma| = 2\cdot 2$ because of the action of the hyperelliptic involutions on the two boundary components.
- $q_1 = 0$, $q_2 = 1$, $m = 2$
- $\dim_{\mathbb{C}}\mathcal{H}(k_i-1) = k_i+1$

Applying formula (4.12) we get:

$$c_{area}(\mathcal{C}_1) = 4^2k_1k_2\frac{1}{2^6}\frac{k_1!k_2!\operatorname{Vol}\mathcal{H}_{1/2}^{hyp}(k_1-1)\operatorname{Vol}\mathcal{H}_{1/2}^{hyp}(k_2-1)}{(k_1+k_2+3)!\operatorname{Vol}\mathcal{Q}_1^{hyp}(k_1^2,k_2^2)}$$

Plugging values (4.9) and (4.1) of volumes we obtain (4.13).

Taking care of the numbering of the zeroes, there is only one such configuration.

b. Configuration $\mathcal{C}_1$ for $k_1 \geq 1, k_2 = -1$: it is a degeneration of the first configuration:

- $M_c = 4^2$, $M_t = 1$

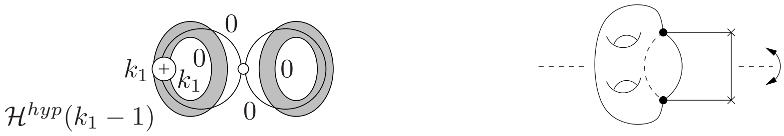


Figure 4.3: Configuration $\mathcal{C}_1^{deg}$ for $\mathcal{Q}^{hyp}(k_1^2, (-1)^2)$

- $M_s = \dfrac{2k_1}{|\Gamma|}$, $|\Gamma| = 2$ (hyperelliptic involution)
- $q_1 = 0$, $q_2 = 1$, $m = 1$
- $\dim_{\mathbb{C}} \mathcal{H}(k_i - 1) = k_i + 1$

Applying formula (4.12) we get:

$$c_{area}(\mathcal{C}_1^{deg}) = 4^2 \frac{2k_1}{2} \frac{1}{2^5} \frac{k_1! \operatorname{Vol} \mathcal{H}_{1/2}^{hyp}(k_1 - 1)}{(k_1 + 2)! \operatorname{Vol} \mathcal{Q}_1^{hyp}(k_1^2, (-1)^2)}$$

Plugging values (4.9) and (4.1) of volumes we obtain

$$c_{area}(\mathcal{C}_1^{deg}) = \frac{k_1 + 3}{(k_1 + 2) 4\pi^2}$$

which is equivalent to (4.13).

c. Configuration $\mathcal{C}_2(k_i)$ for $k_i \geq 1$:

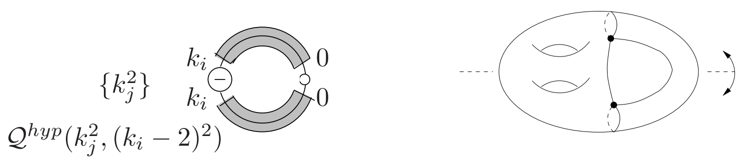


Figure 4.4: Configuration $\mathcal{C}_2$ for $\mathcal{Q}^{hyp}(k_1^2, k_2^2)$

- $M_c = 4^2$, $M_t = 1$
- $M_s = k_i$. Here once we have chosen a direction in which we make the surgery for one of the two zeroes of order $k_i - 2$ of the boundary surface, the direction for the other zero is determined by the hyperelliptic involution.
- $q_1 = 1$, $q_2 = 0$, $m = 1$
- $\dim_{\mathbb{C}} \mathcal{Q}(k_j^2, (k_i - 2)^2) = k_1 + k_2 + 2$

Applying formula (4.12) we get:

$$c_{area}(\mathcal{C}_2(k_i)) = 4^2 k_i \frac{4}{2^5} \frac{(k_1 + k_2 + 1)! \operatorname{Vol} \mathcal{Q}_1^{hyp}(k_j^2, (k_i - 2)^2)}{(k_1 + k_2 + 3)! \operatorname{Vol} \mathcal{Q}_1^{hyp}(k_1^2, k_2^2)}$$

Plugging values (4.1) of volumes we obtain (4.14).

For each k_i there is only one such configuration.

Summing on all configurations we find the known value (8) for the entire hyperelliptic component. $\square$

$\mathcal{Q}^{hyp}(k_1^2, 2k_2+2)$ with k_1 odd and k_2 even

Proposition 13. *For the second type of hyperelliptic components, the Siegel–Veech constants for the configurations described on Figure 4.1 are given by the following formulae:*

$$c_{area}(\mathcal{C}_1) = \frac{(k_1+k_2+4)}{(k_1+2)(k_2+2)4\pi^2} \tag{4.15}$$

$$c_{area}(\mathcal{C}_2) = \frac{(k_2+1)(k_1+k_2+4)}{2(k_1+k_2+2)\pi^2} \tag{4.16}$$

$$c_{area}(\mathcal{C}_3) = \frac{(k_1+1)(k_1+k_2+4)}{2(k_1+k_2+2)\pi^2} \tag{4.17}$$

Moreover if $k_2 \neq 0$:

$$c_{area}(\mathcal{Q}^{hyp}(k_1^2, 2k_2+2)) = c_{area}(\mathcal{C}_1) + c_{area}(\mathcal{C}_2) + c_{area}(\mathcal{C}_3).$$

If $k_2 = 0$ and $k_1 \neq -1$, for the additional configuration we have:

$$c_{area}(\mathcal{C}_{add}) = \frac{5(k_1+4)}{8(k_1+2)\pi^2} \tag{4.18}$$

and we have:

$$c_{area}(\mathcal{Q}^{hyp}(k_1^2, 2)) = c_{area}(\mathcal{C}_{add}) + c_{area}(\mathcal{C}_3).$$

For $\mathcal{Q}(2, -1^2)$, cf § 4.2.3.

Example 3.

$$c_{area}(\mathcal{Q}(2, 1^2)) = c_{area}(\mathcal{C}_{add}) + c_{area}(\mathcal{C}_3) = \frac{25}{24\pi^2} + \frac{5}{3\pi^2} = \frac{65}{24\pi^2}$$

Proof. For each configuration on the list (Figure 4.1) we apply the formula (4.12).

a. Configuration $\mathcal{C}_1$:

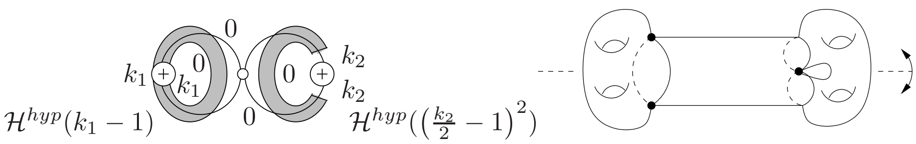


Figure 4.5: Configuration $\mathcal{C}_1$ for $\mathcal{Q}^{hyp}(k_1^2, 2k_2+2)$

- $M_c = 4^2$, $M_t = 2$
- $M_s = k_1 k_2$, same reasons as for the first type.
- $q_1 = 0$, $q_2 = 1$, $m = 2$
- $\dim_{\mathbb{C}} \mathcal{H}(k_1-1) = k_1+1$, $\dim_{\mathbb{C}} \mathcal{H}((\frac{k_2}{2}-1)^2) = k_2+1$

Applying formula (4.12) we get:

$$c_{area}(\mathcal{C}_1) = 4^2 \frac{k_1 k_2}{2} \frac{1}{2^6} \frac{k_1! k_2! \operatorname{Vol} \mathcal{H}_{1/2}^{hyp}(k_1-1) \operatorname{Vol} \mathcal{H}_{1/2}^{hyp}((\frac{k_2}{2}-1)^2)}{(k_1+k_2+3)! \operatorname{Vol} \mathcal{Q}_1^{hyp}(k_1^2, 2k_2+2)}$$

Plugging values (4.9), (4.10) and (4.3) of volumes we obtain (4.15).

There is only one such configuration.

b. Configuration $\mathcal{C}_2$:

Figure 4.6: Configuration $\mathcal{C}_2$ for $\mathcal{Q}^{hyp}(k_1^2, 2k_2 + 2)$

- $M_c = 4^2$, $M_t = 1$
- $M_s = \dfrac{2k_2}{|\Gamma|}$, $|\Gamma| = 2$.
- $q_1 = 1$, $q_2 = 0$, $m = 1$
- $\dim_{\mathbb{C}} \mathcal{Q}(k_1^2, 2k_2 - 2) = k_1 + k_2 + 2$

Applying formula (4.12) we get:

$$c_{area}(\mathcal{C}_2) = 4^2 \frac{2k_2}{2} \frac{4}{2^5} \frac{(k_1 + k_2 + 1)! \operatorname{Vol} \mathcal{Q}_1^{hyp}(k_1^2, 2k_2 - 2)}{(k_1 + k_2 + 3)! \operatorname{Vol} \mathcal{Q}_1^{hyp}(k_1^2, 2k_2 + 2)}$$

Plugging values (4.3) of volumes we obtain (4.16). There is only one configuration of this type.

c. Configuration $\mathcal{C}_3$:

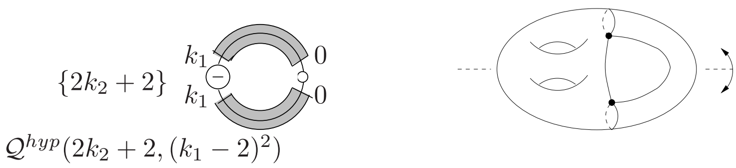


Figure 4.7: Configuration $\mathcal{C}_3$ for $\mathcal{Q}^{hyp}(k_1^2, 2k_2 + 2)$

- $M_c = 4^2$, $M_t = 1$, $M_s = k_1$
- $q_1 = 1$, $q_2 = 0$, $m = 1$
- $\dim_{\mathbb{C}} \mathcal{Q}(2k_2 + 2, (k_1 - 2)^2) = k_1 + k_2 + 2$

Applying formula (4.12) we get:

$$c_{area}(\mathcal{C}_3) = 4^2 k_1 \frac{4}{2^5} \frac{(k_1 + k_2 + 1)! \operatorname{Vol} \mathcal{Q}_1(2k_2 + 2, (k_1 - 2)^2)}{(k_1 + k_2 + 3)! \operatorname{Vol} \mathcal{Q}_1(k_1^2, 2k_2 + 2)}$$

Plugging values (4.3) of volumes we obtain (4.17). There is only one configuration of this type.

Summing on all the three configurations we find the known value (8) for the entire component.

d. For $\mathcal{Q}^{hyp}(k_1^2, 2)$ there is an additional configuration:

- $M_c = 4^3$, $M_t = 2$, $M_s = k_1$
- $q_1 = 1$, $q_2 = 1$, $m = 1$
- $\dim_{\mathbb{C}} \mathcal{H}(k_1 - 1) = k_1 + 1$

Applying formula (4.12) we get:

$$c_{area}(\mathcal{C}_{add}) = 4^3 \frac{k_1}{2} \frac{4+1}{2^6} \frac{k_1! \operatorname{Vol} \mathcal{H}_{1/2}(k_1 - 1)}{(k_1 + 3)! \operatorname{Vol} \mathcal{Q}_1(k_1^2, 2)}$$

Plugging values (4.9) and (4.1) of volumes we obtain (13).

□

Figure 4.8: Additional configuration for $\mathcal{Q}^{hyp}(k_1^2, 2)$

$\mathcal{Q}^{hyp}(2k_1+2, 2k_2+2)$ **with** k_1 **and** k_2 **even**

Proposition 14. *For the third type of hyperelliptic components, the Siegel–Veech constants for the configurations described on Figure 4.1 are given by the following formulae:*

$$c_{area}(\mathcal{C}_1) = \frac{k_1+k_2+4}{4(k_1+2)(k_2+2)\pi^2} \tag{4.19}$$

$$c_{area}(\mathcal{C}_2(k_i)) = \frac{(k_1+k_2+4)(k_i+1)}{2(k_1+k_2+2)\pi^2} \tag{4.20}$$

Moreover:

$$c_{area}(\mathcal{Q}^{hyp}(2k_1+2, 2k_2+2)) = c_{area}(\mathcal{C}_1) + c_{area}(\mathcal{C}_2(k_1)) + c_{area}(\mathcal{C}_2(k_2))$$

If $k_2 = 0$ *and* $k_1 \neq 0$*:*

$$c_{area}(\mathcal{C}_{add}) = \frac{5(k_1+4)}{8(k_1+2)\pi^2} \tag{4.21}$$

and:

$$c_{area}(\mathcal{Q}^{hyp}(2k_1+2, 2)) = c_{area}(\mathcal{C}_2(k_1)) + c_{area}(\mathcal{C}_{add}).$$

For $\mathcal{Q}(2,2)$, cf § 4.2.3.

Proof. For each configuration on the list (Figure 4.1) we apply the formula (4.12).

a. Configuration $\mathcal{C}_1$:

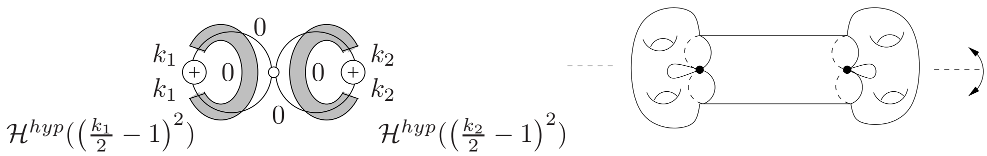


Figure 4.9: Configuration $\mathcal{C}_1$ for $\mathcal{Q}^{hyp}(2k_1+2, 2k_2+2)$

- $M_c = 4^2$, $M_t = 1$, $M_s = k_1k_2$
- $q_1 = 0$, $q_2 = 1$, $m = 2$
- $\dim_{\mathbb{C}} \mathcal{H}((\frac{k_i}{2}-1)) = k_i + 1$

Applying formula (4.12) we get:

$$c_{area}(\mathcal{C}_1) = 4^2 \frac{k_1k_2}{1} \frac{1}{2^6} \frac{k_1!k_2! \operatorname{Vol}\mathcal{H}_{1/2}((\frac{k_1}{2}-1)^2) \operatorname{Vol}\mathcal{H}_{1/2}((\frac{k_2}{2}-1)^2)}{(k_1+k_2+3)! \operatorname{Vol}\mathcal{Q}_1(2k_1+2, 2k_2+2)}$$

Plugging values (4.10) and (4.4) of volumes we obtain (4.19).

b. Configuration $\mathcal{C}_2(k_i)$:

- $M_c = 4^2$, $M_t = 1$, $M_s = k_i$

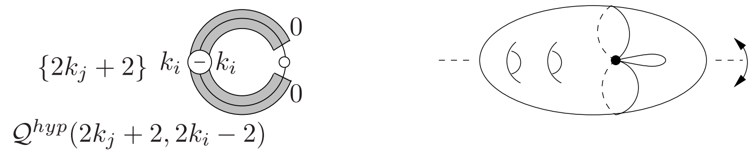


Figure 4.10: Configuration $\mathcal{C}_2$ for $\mathcal{Q}^{hyp}(2k_1+2, 2k_2+2)$

- $q_1 = 1$, $q_2 = 0$, $m = 1$
- $\dim_{\mathbb{C}} \mathcal{Q}(2k_j+2, 2k_i-2) = k_1 + k_2 + 2$

Applying formula (4.12) we get:

$$c_{area}(\mathcal{C}_2) = 4^2 k_i \frac{4}{2^5} \frac{(k_1+k_2+1)! \operatorname{Vol} \mathcal{Q}_1(2k_j+2, 2k_i-2)}{(k_1+k_2+3)! \operatorname{Vol} \mathcal{Q}_1(2k_1+2, 2k_2+2)}$$

Plugging values (4.4) of volumes we obtain (4.20).

c. For $\mathcal{Q}^{hyp}(2k_i+2, 2)$ there is an additional configuration:

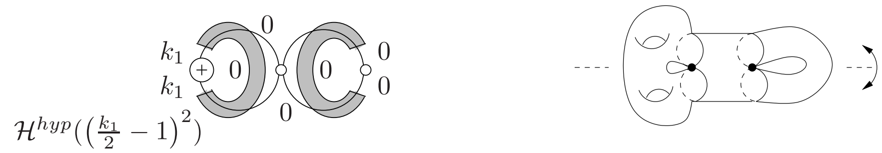


Figure 4.11: Additional configuration for $\mathcal{Q}^{hyp}(2k_1+2, 2)$

- $M = 4^3$, $M_t = 2$, $M_s = k_1$
- $q_1 = 1$, $q_2 = 1$, $m = 1$
- $\dim_{\mathbb{C}} \mathcal{H}((\frac{k_1}{2}-1)^2) = k_1 + 1$

Applying formula (4.12) we get:

$$c_{area}(\mathcal{C}_{add}) = 4^3 \frac{k_1}{2} \frac{5}{2^5} \frac{k_1! \operatorname{Vol} \mathcal{H}_{1/2}((\frac{k_1}{2}-1)^2)}{(k_1+3)! \operatorname{Vol} \mathcal{Q}_1(2k_1+2, 2)}$$

Plugging values (4.9) and (4.1) of volumes we obtain (14).

□

4.2.3 Special cases: empty boundary stratum

The strata $\mathcal{Q}(2, -1^2)$ and $\mathcal{Q}(2, 2)$ have no boundary stratum (cf §3.3.3), so their configurations are degenerations of the configurations presented on the previous paragraph. With $\mathcal{Q}(-1^4)$ they are the only strata with an empty boundary stratum.

Note that furthermore they are connected.

Stratum $\mathcal{Q}(2, -1^2)$

This stratum is hyperelliptic of second type with $k_2 = 0$ and $k_1 = -1$.

Recall the value of the volume computed in (4.6):

$$\operatorname{Vol} \mathcal{Q}_1^{num}(2, -1^2) = \frac{4}{3}\pi^2.$$

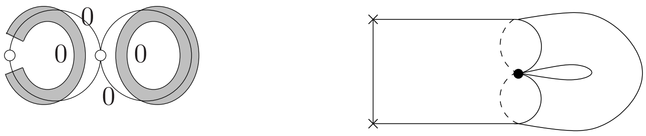


Figure 4.12: The only configuration of $\mathcal{Q}(2,-1,-1)$ containing cylinders, and its topological picture

For an alternative computation of this volume using graphs, see section 4.7.1.

The only one configuration is a degeneration of the configuration of Figure 4.8, shown in Figure 4.12.

We have the following configuration data:

- $M_c = 4^3$, $M_t = 2$
- $q_1 = 1 = q_2$

In this case we have to use a variation of the main formula, given by equation (3.16), we obtain:

$$\begin{aligned} c_{area}(\mathcal{C}) &= \frac{1}{2} 4^3 \frac{4+1}{2^5 (2)! \operatorname{Vol} \mathcal{Q}_1(2,-1^2)} \\ &= \frac{5}{2 \operatorname{Vol} \mathcal{Q}_1(2,-1^2)} \end{aligned}$$

The configuration counts only once, so with the computed value of the volume it gives:

$$c_{area}(\mathcal{Q}_1(2,-1^2)) = \frac{15}{8\pi^2}$$

which is the known value of $c_{area}(\mathcal{Q}(2,-1^2))$

Stratum $\mathcal{Q}(2,2)$

This stratum is hyperelliptic of the third type with $k_1 = k_2 = 0$.

Recall the value of the volume computed in (4.7):

$$\operatorname{Vol} \mathcal{Q}_1^{num}(2,2) = \frac{4}{3}\pi^2.$$

There is only one configuration shown on Figure 4.13.

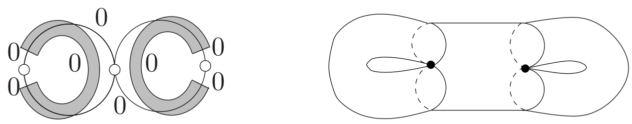


Figure 4.13: Configuration $\mathcal{C}$ for $\mathcal{Q}(2,2)$

Data:

- $M_c = 4^4$, $M_t = 2$
- $q_1 = 2$, $q_2 = 1$
- $\dim_{\mathbb{C}} \mathcal{Q}(2,2) = 4$

Equation (3.16) gives:

$$\begin{aligned} c_{area}(\mathcal{C}) &= \frac{1}{2} 4^4 \frac{4\cdot 2+1}{2^6 (3)! \operatorname{Vol} \mathcal{Q}(2,2)} \\ &= \frac{3}{\operatorname{Vol} \mathcal{Q}_1(2,2)} \end{aligned}$$

The configuration counts only once, so with the computed value of the volume it gives:

$$c_{area}(\mathcal{Q}(2,2)) = \frac{9}{4\pi^2}$$

which is the known value for the stratum.

4.3 Volumes of strata of complex dimension smaller than 5

For strata of complex dimension $d \leq 5$, we can use the ideas developed by Eskin and Okounkov in [EOk1], Zorich in [Zo4], Athreya Eskin and Zorich in [AEZ2], and compute volumes by counting "integer points" in the stratum.

The relation between volume and number of "integer points" is given in § 2.3 of [AEZ2]:

Proposition 15 (Athreya-Eskin-Zorich)**.**

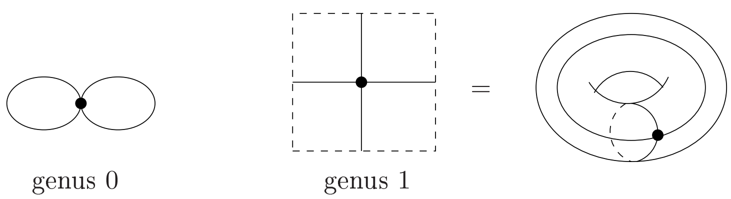
$$\begin{aligned} \operatorname{Vol} \mathcal{Q}_1(\alpha) &= 2d \cdot \lim_{N\to\infty} N^{-d} \cdot \\ & \quad (\textit{Number of "integer points" of area at most } N/2 \textit{ in } \mathcal{Q}(\alpha)) \end{aligned} \tag{4.22}$$

Here we recall briefly the techniques of Athreya, Eskin and Zorich to count integer points (or square-tiles surfaces, or pillowcase covers) in genus 0, and explain how generalize them to genus $g > 0$.

A flat surface (S,ω) corresponding to an integer point, i.e. a point in the lattice $(H_1^-(\hat{S},\hat{\Sigma};\mathbb{Z}))^*_{\mathbb{C}}$ in local coordinates, can be decomposed into horizontal cylinders with half-integer or integer widths, with zeroes and poles lying on the boundaries of these cylinders, that are called singular layers in [AEZ2]. Each layer defines a ribbon graph (graph with a tubular neighbourhood inside the surface), called *map* in combinatorics. A zero of order d_i belonging to a layer corresponds to a vertex of valency $d_i + 2$ in the associated graph, and edges of the graph emerging from this vertex correspond to horizontal rays emerging from the zero in the surface. The graph is metric: edges have half-integer lengths. A ribbon graph or a map carries naturally a genus: it is the minimal genus of the surface in which it can be embedded. So a ribbon graph associated to a singular layer in S has a genus lower or equal to the genus g of S. Also a ribbon graph has some faces corresponding to the connected components of its complementary in the minimal surface in which it can be embedded. In our case faces correspond to cylinders emerging from the layer. In genus 0 each face corresponds to a distinct cylinder, in higher genus some cylinders may have both of their boundaries glued to the same layer. For a ribbon graph Γ we have the Euler relation:

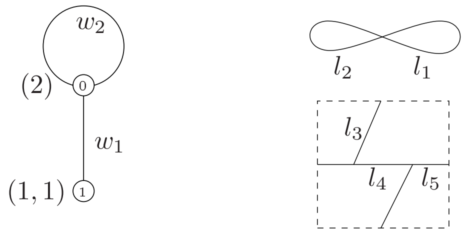
$$\chi_\Gamma = 2 - 2g_\Gamma = V_\Gamma - E_\Gamma + F_\Gamma$$

where g_Γ is the genus of Γ, V_Γ, E_Γ and F_Γ the number of respectively vertices, edges and faces of Γ. In the figure below we represent the two maps with one 4-valent vertex: one is of genus 0 and has 3 faces, the other is of genus 1 and has 1 face.



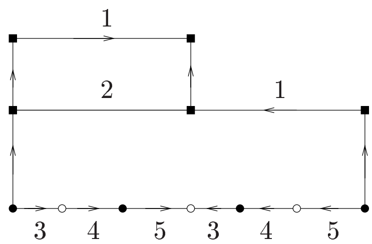
We encode the decomposition of the surface S into horizontal cylinders in a supplementary graph T, by representing each singular layer by a point in this graph and each cylinder emerging form a layer by an edge emerging form the corresponding vertex. So a layer with k faces corresponds to a k-valent vertex in T. We record also the information on the order of the zeroes lying in each layer, and on the genus of the ribbon graph: that gives a decoration of the graph T. For surfaces S of genus 0 this graph is a tree.

As an example we consider a surface in $\mathcal{Q}(2,1^2)$ represented by the following graph:



On the left we figure the graph T. The lower vertex represents a ribbon graph of genus 1 with two zeroes of order 1 (two 3-valent vertices): the corresponding layer is drawn on the right. The higher vertex corresponds to the ribbon graph of genus 0 with one 4-valent vertex (zero of order 2) drawn on the right. The width w_i of the cylinders and the lengths l_i of the edges of the ribbon graphs are also recorded.

Below is a flat representation of a surface corresponding to the configuration described above.



Note that the genus of S is the sum of the genera of the vertices of T, and the genus created by loops in the graph T: namely, the dimension of the homology of the graph T. In the example, the surface is of genus 2.

Note also that horizontal cylinders in S which are homologous to 0 correspond to separating edges on the graph T. It will be useful because with Convention 2, the width w of a cylinder is an integer if its waist curve is homologous to 0, and half-integer otherwise. In the example w_1 is integer and w_2 half-integer (furthermore here w_1 is necessarily equal to $2w_2$).

We have to choose the l_i such that the length of the boundaries of the faces of the ribbon graphs Γ_j correspond to the w_k. In the example we have necessarily $w_2 = l_1 = l_2$ and $w_1 = 2l_1 = 2w_1 = 2(l_3 + l_4 + l_5)$. So we have only one choice for l_1 and l_2 and exactly $\sum_{i=1}^{w_1-2}(i-1) = \frac{(w_1-2)^2}{2}$ choices for (l_3, l_4, l_5) (see also Lemma 9), because with the convention 2, w_2 is an integer and the l_i are half-integer.

To count surfaces of area lower than $N/2$ corresponding to lattice points, we have to sum on the possible graphs T and on the possible corresponding layers Γ, the number of distinct flat surfaces of this combinatorial type. So for a fixed graph T and fixed layers Γ_i we have to count the number of twists t_j, widths w_i, heights h_i and lengths of saddle connexions l_i satisfying the combinatorial configuration, and such that the area $w \cdot h = \sum_i w_i h_i$ is lower or equal to $N/2$. More precisely by (4.22) we have to get the asymptotic of this number as N goes to infinity. In the example all the l_i are half-integer, h_1, t_1, h_2, t_2 also because they are coordinates of saddle connexions that are non homologous to zero, w_2 is half-integer and w_1 is integer. Twists t_1 and t_2 take respectively $2w_1$ and $2w_2$ half-integer values. We have already seen that the l_i take $\frac{(w_1-2)^2}{2}$ values (with the condition $w_1 = 2w_2$). So we want to find the asymptotic of

$$\sum_{\substack{w_1h_1+w_2h_2\leq N/2\\ w_1\in\mathbb{N},\\ w_2,h_1,h_2\in\mathbb{N}/2}} 2w_1 2w_2 \frac{(w_1-2)^2}{2}\mathbb{1}_{\{w_1=2w_2\}} = \sum_{\substack{w(h_1+2h_2)\leq N/2\\ w\in\mathbb{N},h_1,\\ h_2\in\mathbb{N}/2}} 8w^2\frac{(2w-2)^2}{2}$$

Remark that since we want only the term of higher order in N we just need to take the term of higher order in w_i, so we can replace $\frac{(2w-2)^2}{2}$ by $\frac{(2w)^2}{2} = 2w^2$. In general the asymptotic for such sums is given by Lemma 3.7 of [AEZ2]. For this particular case, it is given by Lemma 11, and we obtain $\frac{N^5}{10}(32\zeta(4) - 33\zeta(5))$.

This approach is somehow limited because we need to known all the ribbon graphs of a certain type and the number of these ribbon graphs increases fast as the dimension of the stratum grows.

So we apply this method to strata of complex dimension $d \leq 5$, using the complete description of ribbon graphs with at most 5 edges given in [JV]: recall that a zero of order d_i corresponds in the ribbon graph to a vertex with $d_i + 2$ adjacent edges, so the maximal number of edges of a ribbon graph in a strata $\mathcal{Q}(d_1, \ldots, d_n)$ is

$$\frac{\sum_{i=1}^n d_i}{2} = 2g - 2 + n = \dim_{\mathbb{C}} \mathcal{Q}(d_1, \ldots, d_n).$$

In genus 0, Athreya, Eskin and Zorich were able to compute the volumes of all strata of type $\mathcal{Q}(1^K, -1^{K+4})$ with this method because they used a formula which gives directly the number of ways the cylinders of widths w_i can be glued at a vertex j of a tree T. This formula was deduced from a formula of M. Kontsevich by a recurrence on the number of poles. The formula of Kontsevich works also for higher genus, but for distinct widths w_i, and since cylinders can form some loops in the surface, it is not obvious to get a general formula for the higher genus case, even for the strata $\mathcal{Q}(1^k, -1^l)$.

Convention 4. In the following we write the half-integers in lower case and the integers in capitals.

4.4 First example: $\mathcal{Q}(5,-1)$

For this stratum, the Siegel–Veech constant $c_{area}(\mathcal{Q}(5,-1))$ is known, cf Theorem 9.1. in [CM]. We compute the volume of this stratum following the method described in section 4.3. We evaluate also all combinatorial parameters appearing in (3.1), and so we check formula (3.1) in this case.

4.4.1 Volume of $Q(5,-1)$

We use here the method described in § 4.3 to compute by hands the volume of $\mathcal{Q}(5,-1)$. In this case, there are only two possible graphs T, and for each graph, only two possible layers. This gives four configurations (note that here we do not speak about configurations of $\hat{\text{h}}$omologous cylinders, but about configurations of horizontal cylinders for integer surfaces in the stratum). The computations of the asymptotics are detailed in the section 4.8.

- Configuration 1:

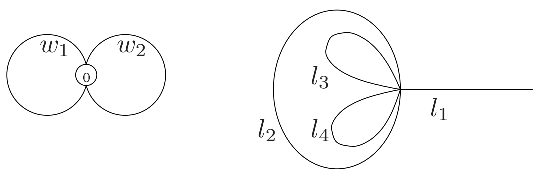


Figure 4.14: Configuration 1

Convention 2 implies that all parameters w_i, h_i, t_i, l_i are half-integers. The possible lengths of the waist curves of the cylinders are l_3, l_4, $l_2 + 2l_1$ and $l_2 + l_3 + l_4$. Since $l_2 + l_3 + l_4 > l_3$ and $l_2 + l_3 + l_4 > l_4$ we should have $l_3 = l_4$ and $l_2 + 2l_1 = l_2 + 2l_3$:

$$\begin{cases} w_1 = l_3 = l_4 \\ w_2 = l_2 + 2l_1 = l_2 + 2l_3 \end{cases}$$

There is one way to find such (l_1, l_2, l_3, l_4), if $2w_1 < w_2$. The contribution to the counting for this configuration is:

$$\sum_{(w_1h_1+w_2h_2)\leq N/2} 4w_1w_2(\mathbb{1}_{\{2w_1<w_2\}}) = \sum_{(W_1H_1+W_2H_2)\leq 2N} W_1W_2(\mathbb{1}_{\{2W_1<W_2\}})$$

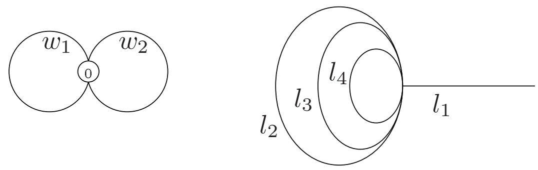


Figure 4.15: Configuration 2

- Configuration 2:

All parameters are half-integers. The possible lengths for the waist curves of the cylinders are l_4, l_3+l_4 , l_2+l_3 and l_2+2l_1. Since $l_3+l_4>l_4$ and the situation

$$\begin{cases} l_4 = l_2+2l_1 \\ l_3+l_4 = l_2+l_3 \end{cases}$$

is impossible, the only remaining case is:

$$\begin{cases} w_1 = l_4 = l_2+l_3 \\ w_2 = l_3+l_4 = l_2+2l_1 \end{cases} .$$

This implies that $l_3 = l_1$ and there is only one way to find such l_i, but only if $w_1 < w_2 < 2w_1$. The contribution to the counting is:

$$\sum_{(w_1h_1+w_2h_2)\le N/2} 4w_1w_2(\mathbb{1}_{\{w_1<w_2<2w_1\}}) = \sum_{(W_1H_1+W_2H_2)\le 2N} W_1W_2(\mathbb{1}_{\{W_1<W_2<2W_1\}})$$

Summing the contributions of the 2 first configurations gives:

$$\begin{aligned} \sum_{(W\cdot H)\le 2N} W_1W_2(\mathbb{1}_{\{2W_1<W_2\}}+\mathbb{1}_{\{W_1<W_2<2W_1\}}) &= \sum_{W.H\le 2N} W_1W_2\mathbb{1}_{\{W_1<W_2\}} \\ &\sim \frac{1}{2}\frac{(2N)^4}{4!}(\zeta(2))^2 = \frac{N^4(\zeta(2))^2}{3} \end{aligned}$$

- Configuration 3:

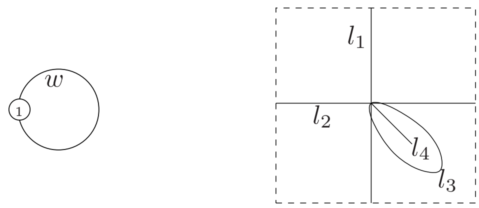


Figure 4.16: Configuration 3

All parameters are half-integers. The two lengths are $2l_1+2l_2+l_3$ and l_3+2l_4 so we should have $l_4 = l_2+l_1$ in order that the two are equal. Then we search the number of (l_1, l_2, l_3) such that $w = l_3+2(l_1+l_2)$. It is a polynomial of w with leading term $\frac{1}{4}\frac{(2w)^2}{2} = \frac{w^2}{2}$.
The contribution to the counting is:

$$\sum_{wh\le N/2} 2\frac{w^3}{2} = \sum_{WH\le 2N}\left(\frac{W}{2}\right)^3 \sim \frac{1}{8}\frac{(2N)^4}{4}\zeta(4) = \frac{\zeta(4)}{2}N^4$$

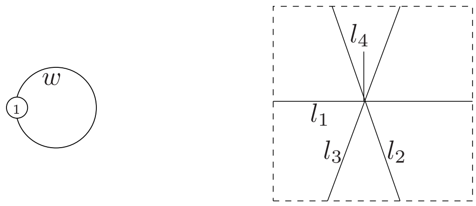


Figure 4.17: Configuration 4

- Configuration 4:

  All parameters are half integers. The lengths for the waist curves are $2l_1 + l_2 + l_3$ and $2l_4+l_2+l_3$, so we have $l_1 = l_4$. The number of solutions of $w = 2l_1+l_2+l_3$ is approximately $\frac{1}{2}\frac{(2w)^2}{2} = w^2$.

  The contribution to the counting for this configuration:

$$\sum_{wh\leq N/2} 2w^3 = \sum_{WH\leq 2N} 2\left(\frac{W}{2}\right)^3 = 2\frac{1}{8}\frac{(2N)^4}{4}\zeta(4) = \zeta(4)N^4.$$

- Total:

  The sum of the 4 contributions is:

$$N^4\left(\frac{(\zeta(2))^2}{3} + \frac{3}{2}\zeta(4)\right) = \frac{7\pi^4 N^4}{2\cdot 3^3\cdot 5}$$

  We obtain:

$$\operatorname{Vol} Q(5,-1) = \dim_{\mathbb{R}} Q(5,-1)\frac{7}{2\cdot 3^3\cdot 5}\pi^4 = \frac{2^2\cdot 7}{3^3\cdot 5}\pi^4$$

4.4.2 Siegel–Veech constant

For this strata, the value of the Siegel-Veech constant is known:

$$c_{area}Q(5,-1) = \frac{15}{7\pi^2}.$$

There is only one configuration $\mathcal{C}$ containing cylinders, cf Figure 4.18. To understand how to read the graph corresponding to the configuration, see § 3.2.3.

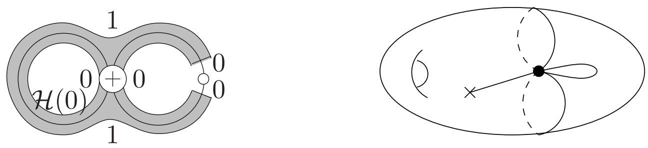


Figure 4.18: The only configuration of $\mathcal{Q}(5,-1)$ containing cylinders, and its topological picture

For this configuration, we take the family of curves as in § 3.3.2, cf Figure 4.19. We evaluate the combinatorial constants that appear in (3.17).

First we can see that all cycles in the family $\{\gamma,\delta\}$ are not homologous to zero in $H_1(S,\{P_i\},\mathbb{Z})$. Thus $M_c = 4^2$.

There is only one thin cylinder so $M_t = 1$.

The surgery applied to the marked point on the torus belonging to the principal boundary stratum $\mathcal{H}(0)$ is of type $+4.1a$ (local construction). There are 2 geodesics rays coming from this point. But choosing either of the geodesic rays do not change the configuration because the involution of the torus exchanges these two rays, so $M_s = \frac{2}{|\Gamma|} = 1$.

We have the following combinatorial data:

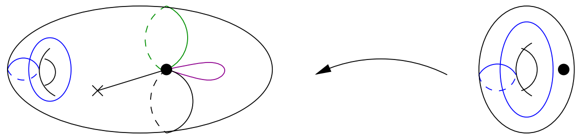


Figure 4.19: Family of curves associated to the configuration

- $\dim_{\mathbb{C}} \mathcal{H}(0) = 2$,
- $\dim_{\mathbb{C}} \mathcal{Q}(5,-1) = 4$,
- $\operatorname{Vol} \mathcal{H}_1(0) = \frac{\pi^2}{3}$,
- $q_1 = 1$, $q_2 = 0$.

Applying formula (3.17) we get :

$$c_{area}(\mathcal{C}) = 4^2 \frac{4}{2^5} \frac{(2-1)! 2^2 \pi^2/3}{(4-1)! \operatorname{Vol} \mathcal{Q}(5,-1)}.$$

So:

$$c_{area}(\mathcal{C}) = \frac{2^2 \pi^2}{3^2 \operatorname{Vol} \mathcal{Q}_1(5,-1)}$$

The value of the volume computed in the previous section is $\operatorname{Vol} \mathcal{Q}_1(5,-1) = \frac{2^2 \cdot 7}{3^3 \cdot 5} \pi^4$, it gives:

$$c_{area}(\mathcal{C}) = \frac{15}{7\pi^2},$$

which coincides with value of Theorem 9.1. in [CM].

4.5 Second example: $\mathcal{Q}(3,-1^3)$

This stratum is also non-varying so the Siegel–Veech constant for the entire stratum is known (cf [CM]), and it is of complex dimension 4, which allows a computation of the volumes by counting graphs.

4.5.1 Volume

As previously we compute the volume of this stratum using the method described in § 4.3.

- Configuration 1

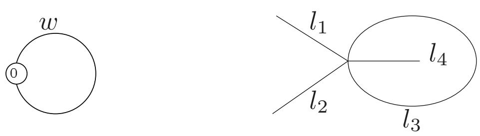


Figure 4.20: Configuration 1

All parameters are half-integers. The constraints are given by: $w = l_3 + 2l_4 = l_3 + 2l_1 + 2l_2$. There are $\sim \frac{1}{4}\frac{(2w)^2}{2} = \frac{w^2}{2}$ choices for the l_i. There are 6 ways to give name to the poles. The contribution to the counting is

$$6 \sum_{w.h \leq N/2} 2w \frac{w^2}{2} = 6 \sum_{WH \leq 2N} \left(\frac{W}{2}\right)^3 \sim \frac{3}{4}\frac{(2N)^4}{4}\zeta(4) = 3\zeta(4) N^4$$

- Configuration 2

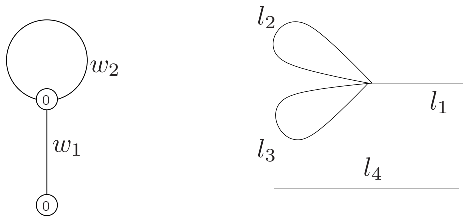


Figure 4.21: Configuration 2

The parameter $w_1 = W_1$ is an integer and all remaining parameters are half-integers. Note that here there are 3 ways to give names to the poles. The equations

$$\begin{cases} w_2 = l_2 = l_3 \\ W_1 = 2l_1 + l_2 + l_3 = 2l_4 \end{cases}$$

have one solution if $W_1 > 2w_2$.

The contribution of this configuration is:

$$3 \sum_{W_1 h_1 + w_2 h_2 \leq N/2} 2w_1 2w_2 \mathbb{1}_{\{W_1 > 2w_2\}} = 6 \sum_{2W_1 H_1 + W_2 H_2 \leq 2N} W_1 W_2 \mathbb{1}_{\{W_1 > W_2\}}$$

- Configuration 3

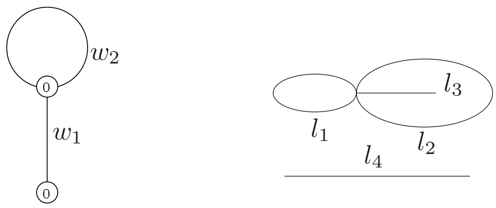


Figure 4.22: Configuration 3

The parameter $w_1 = W_1$ is an integer and all remaining parameters are half-integers. Note that here there are 3 ways to give names to the poles.

Two ribbon graphs are possible for the second layer:

For the first ribbon graph, the equations

$$\begin{cases} W_1 = 2l_4 = l_1 + l_2 \\ w_2 = l_1 = l_2 + 2l_3 \end{cases}$$

have one solution if $w_2 < W_1 < 2w_2$.

For the second ribbon graph, the equations

$$\begin{cases} W_1 = 2l_4 = l_1 \\ w_2 = l_2 + 2l_3 = l_2 + l_1 \end{cases}$$

have one solution if $W_1 < w_2$.

The total number of solutions is then:

$$\mathbb{1}_{\{w_2 < W_1 < 2w_2\}} + \mathbb{1}_{\{W_1 < w_2\}} = \mathbb{1}_{\{W_1 < 2w_2\}} - \underbrace{\mathbb{1}_{\{W_1 = w_2\}}}_{negligible}$$

This gives a contribution:

$$3 \sum_{W_1h_1+w_2h_2 \leq N/2} 2w_1 2w_2 \mathbb{1}_{\{W_1<2w_2\}} = 6 \sum_{2W_1H_1+W_2H_2 \leq 2N} W_1W_2 \mathbb{1}_{\{W_1<W_2\}}$$

Summing the contributions of configurations 2 and 3 we get:

$$6 \sum_{2W_1H_1+W_2H_2 \leq 2N} W_1W_2 = \frac{1}{4} 6 \sum_{W \cdot H \leq 2N} W_1W_2 \sim \frac{3}{2}\frac{(2N)^4}{4!}(\zeta(2))^2 = \frac{5N^4}{2}\zeta(4)$$

- Configuration 4

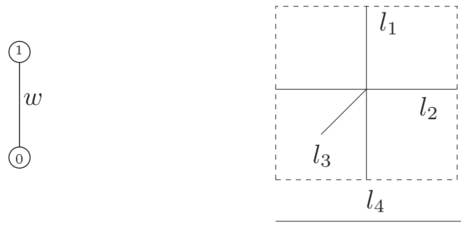


Figure 4.23: Configuration 4

The parameter $w = W$ is an integer and all remaining parameters are half-integers. Note that here also there are 3 ways to give name to the poles.

The constraints are:

$$W = 2l_4 = 2(l_1 + l_2 + l_3)$$

So there are $\sim \frac{W^2}{2}$ ways to choose $(l_1, \ldots, l_4)$.

The contribution of this configuration is:

$$3 \sum_{W.h \leq N/2} 2W \frac{W^2}{2} = 3 \sum_{WH \leq N} W^3 \sim \frac{3N^4}{4}\zeta(4)$$

- The sum of all contributions is $\frac{25N^4}{4}\zeta(4)$ so it gives

$$\mathrm{Vol}^{comp}\, \mathcal{Q}(3, -1^3) = 50\zeta(4) = \frac{5\pi^4}{9}$$

4.5.2 Siegel–Veech constant

The stratum $\mathcal{Q}(3, -1^3)$ is non-varying, so the value of the sum of the Lyapunov exponents λ^+ of the invariant subbundle H^1_+ of the Hodge bundle over the stratum along the Teichmüller flow is given in [CM]: $L^+ = \frac{2}{5}$. By the Eskin-Kontsevich-Zorich formula ([EKZ2]) we obtain

$$c_{area}(\mathcal{Q}(3, -1^3)) = \frac{9}{5\pi^2}.$$

There are two configurations for this stratum.

a. Configuration $\mathcal{C}_1$:

The combinatorial data for this configuration are:

- $M_c = 4^2$, $M_s = 1$, $M_t = 1$
- $q_1 = 1$, $q_2 = 0$
- $\dim_{\mathbb{C}} \mathcal{Q}(3, -1^3) = 4$
- $\dim_{\mathbb{C}} \mathcal{Q}(-1^4) = 2$.
- $\mathrm{Vol}\, \mathcal{Q}_1(-1^4) = 2\pi^2$ ([AEZ1])

Figure 4.24: Configuration $\mathcal{C}_1$ of $\mathcal{Q}(3,-1,-1,-1)$ containing cylinders, and its topological picture

Applying formulae of Theorem 8 gives:

$$c_{area}(\mathcal{C}_1) = \frac{2\pi^2}{3\,\mathrm{Vol}\,\mathcal{Q}_1(3,-1^3)}$$

There is only one configuration of type $\mathcal{C}_1$.

b. Configuration $\mathcal{C}_2$:

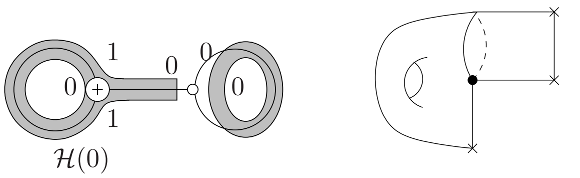


Figure 4.25: Configuration $\mathcal{C}_2$ of $\mathcal{Q}(3,-1,-1,-1)$ containing cylinders, and its topological picture

The combinatorial data are:

- $M_c = 4^2$, $M_s = \frac{2}{2}$ (hyperelliptic involution), $M_t = 1$.
- $q_1 = 0$, $q_2 = 1$
- $\dim_{\mathbb{C}} \mathcal{H}(0) = 2$.
- $\mathrm{Vol}\,\mathcal{H}_{1/2}(0) = \frac{4\pi^2}{3}$

Applying formulae of Theorem 8 gives:

$$c_{area}(\mathcal{C}_2) = \frac{\pi^2}{9\,\mathrm{Vol}\,\mathcal{Q}_1(3,-1^3)}$$

Due to the numbering of the poles there are 3 configurations of type $\mathcal{C}_2$.

Substituting the computed value of the volume and summing on all configurations we find

$$c_{are}(\mathcal{Q}(3,-1^3)) = c_{area}(\mathcal{C}_1) + 3c_{area}(\mathcal{C}_2) = \frac{6}{5\pi^2} + 3 \cdot \frac{1}{5\pi^2} = \frac{9}{5\pi^2}$$

which is coherent with the formula of Theorem 8.1. in [CM].

4.6 Summary

In the following tabular we gather the data about the strata studied in this paper. We denote L^+ (resp. L^-) the sum of the Lyapunov exponents λ^+ (resp. λ^-) of the invariant subbundle H^1_+ (resp. H^1_-) of the Hodge bundle over the stratum $\mathcal{Q}(d_1,\dots,d_n)$ along the Teichmüller flow.

Denoting

$$K = \frac{1}{24}\sum_{j_1}^{n} \frac{d_j(d_j+4)}{d_j+2} \quad \text{and} \quad I = \frac{1}{4}\sum_{d_j\ odd} \frac{1}{d_j+2},$$

Theorem 2 of [EKZ2] gives

$$\begin{aligned} L^+ &= K + \frac{\pi^2}{3} c_{area}(\mathcal{Q}(\alpha)) \\ L^- - L^+ &= I. \end{aligned}$$

So c_{area} is given by the formula

$$c_{area}(\mathcal{Q}(d_1, \ldots, d_n)) = \frac{3}{\pi^2}(L^+ - K) = \frac{3}{\pi^2}(L^- - I - K).$$

We denote also $g_{eff} = \hat{g} - g$ where $\hat{g}$ is the genus of the double cover surface $\hat{S}$ for $S \in \mathcal{Q}(\alpha)$. Recall that $H^1_-(\hat{S})$ has dimension $2g_{eff}$. In comparison to the Abelian case we expect that the values of the volumes of the strata of quadratic differentials are given by $r\pi^{2g_{eff}}$ with $r \in \mathbb{Q}$. This is true for the examples cited below.

strata	g	g_{eff}	$\dim_{\mathbb{C}}$	K	I	L^+	L^-	c_{area}	boundary comp.	hyp	Vol
$\mathcal{Q}(2,-1^2)$	1	1	3	$-\frac{1}{8}$	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{15}{8\pi^2}$	$\emptyset$	$\mathcal{Q}(0,-1^4)$	$\frac{4}{3}\pi^2$
$\mathcal{Q}(1^2,-1^2)$	1	2	4	$-\frac{1}{9}$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{4}{3}$	$\frac{7}{3\pi^2}$	$\mathcal{H}(0), \mathcal{Q}(-1^4)$	$\mathcal{Q}(1,-1^5)$	$\frac{1}{3}\pi^4$
$\mathcal{Q}(2,2)$	2	1	4	$\frac{1}{4}$	0	1	1	$\frac{9}{4\pi^2}$	$\emptyset$	$\mathcal{Q}(-1^4,0^2)$	$\frac{4}{3}\pi^2$
$\mathcal{Q}(2,1^2)$	2	2	5	$\frac{19}{72}$	$\frac{1}{6}$	$\frac{7}{6}$	$\frac{4}{3}$	$\frac{65}{24\pi^2}$	$\mathcal{Q}(2,-1^2), \mathcal{H}(0)$	$\mathcal{Q}(1,-1^5,0)$	$\frac{2}{15}\pi^4$
$\mathcal{Q}(1^4)$	2	3	6	$\frac{5}{18}$	$\frac{1}{3}$	$\frac{4}{3}$	$\frac{5}{3}$	$\frac{19}{6\pi^2}$	$\mathcal{Q}(1^2,-1^2), \mathcal{H}(0)$	$\mathcal{Q}(1^2,-1^6)$	$\frac{1}{15}\pi^6$
$\mathcal{Q}(5,-1)$	2	2	4	$\frac{1}{7}$	$\frac{2}{7}$	$\frac{6}{7}$	$\frac{9}{7}$	$\frac{15}{7\pi^2}$	$\mathcal{H}(0)$	no	$\frac{28}{135}\pi^4$
$\mathcal{Q}(3,-1^3)$	1	2	4	$-\frac{1}{5}$	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{6}{5}$	$\frac{9}{5\pi^2}$	$\mathcal{Q}(-1^4), \mathcal{H}(0)$	no	$\frac{5}{9}\pi^4$

4.7 Alternative computations of volumes

Here we use the method of § 4.3 and the lemmas of section 4.8 to compute some volumes of hyperelliptic strata already computed in § 4.1. This allows us to check one more time that our choices of normalization for the volumes are consistent.

4.7.1 $\mathcal{Q}(2,-1^2)$

- Configuration 1:

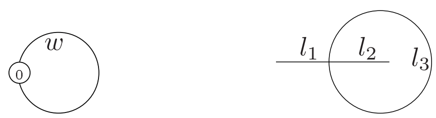


Figure 4.26: Configuration 1

All parameters are half-integers. We have $w = 2l_1 + l_3 = 2l_2 + l_3$, which has $\simeq \frac{2w}{2} = w$ solutions. The contribution is therefore:

$$\sum_{wh \leq N/2} 2w^2 = \sum_{WH \leq 2N} \frac{W^2}{2} \sim \frac{1}{2}\frac{(2N)^3}{3}\zeta(3) = \frac{4N^3}{3}\zeta(3)$$

- Configuration 2:

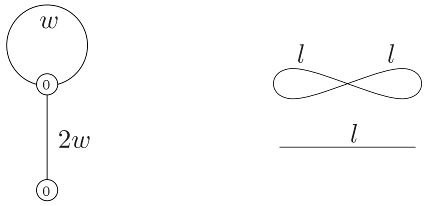


Figure 4.27: Configuration 2

All parameters are half-integers. Equation $w = l$ has 1 solution. We have an additional factor $\frac{1}{o_t} = \frac{1}{2}$ which comes from the definition of the twist (cf § 3.3.3). By Lemma 11 we obtain the following contribution:

$$\frac{1}{2} \sum_{w(2h_1+h_2) \leq N/2} 2w(4w) = \sum_{W(2H_1+H_2) \leq 2N} W^2 \sim \frac{N^3}{6}(8\zeta(2) - 9\zeta(3))$$

- Configuration 3:

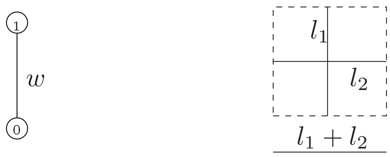


Figure 4.28: Configuration 3

$w = W$ is an integer and l_1, l_2, h, t are half-integers. Equation $W = 2(l_1 + l_2)$ has approximately W solutions. There is an additional factor $\frac{1}{2}$ for the twist, and another factor $\frac{1}{2}$ because (l_2, l_1) and (l_1, l_2) give the same surfaces. The contribution of this configuration is then:

$$\frac{1}{4} \sum_{Wh \leq N/2} 2W \cdot W = \frac{1}{2} \sum_{WH \leq N} W^2 \sim \frac{N^3}{6}\zeta(3)$$

Summing all the contributions we get $\dfrac{4N^3}{3}\zeta(2)$ so by (4.22), we obtain:

$$Vol\mathcal{Q}(2, -1^2) = 8\zeta(2) = \frac{4\pi^2}{3},$$

which coincides with the value found in (4.6).

4.7.2 $\mathcal{Q}(1^2, -1^2)$

- Configuration 1:

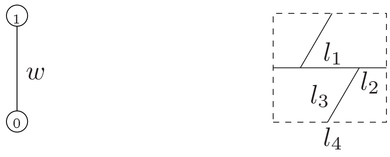


Figure 4.29: Configuration 1

$w = W$ is an integer and all remaining parameters are half-integers. Equation $W = 2(l_1 + l_2 + l_3) = 2l_4$ has approximatively $\dfrac{W^2}{2}$ solutions. Here $o_t = 3$ so we have an extra factor $\frac{1}{3}$.

So the contribution of this configuration is:

$$\frac{1}{3}\sum_{Wh\leq N/2} 2W\cdot\frac{W^2}{2}=\frac{1}{3}\sum_{WH\leq N} W^3\sim\frac{N^4}{12}\zeta(4)$$

- Configuration 2:

Figure 4.30: Configuration 2

All parameters are half-integers. Equation $w=2l_1+l_2+l_3=2l_4+l_2+l_3$ has $\sim\frac{1}{2}\frac{(2w)^2}{2}=w^2$ solutions. There are 2 ways to give name to the zeroes. Therefore the contribution of this configuration is:

$$2\sum_{wh\leq N/2} 2w\cdot w^2=2\sum_{WH\leq 2N} W\left(\frac{W}{2}\right)^2\sim\frac{1}{2}\frac{(2N)^4}{4}\zeta(4)=2N^4\zeta(4)$$

- Configuration 3:

Figure 4.31: Configuration 3

All parameters are half-integers. There are two ways to give name to the zeroes. There are 2 ways to numbered the faces of the ribbon graphs: the first gives:

$$\begin{cases} w_1=l_1\\ w_2=l_1+2l_2\end{cases}$$

so there is $\mathbb{1}_{\{w_1<w_2\}}\mathbb{1}_{\{w_2-w_1\in\mathbb{N}\}}$ solution. The second intertwines w_1 and w_2, but w_1 and w_2 play symmetric roles in the graph T. We get the following contribution:

$$2\sum_{w\cdot h\leq N/2} 2w_1 2w_2\mathbb{1}_{\{w_1<w_2\}}\mathbb{1}_{\{w_2-w_1\in\mathbb{N}\}}=\frac{1}{2}\sum_{W\cdot H\leq 2N} W_1W_2\sim\frac{(2N)^4}{4!}(\zeta(2))^2=\frac{5N^4}{6}\zeta(4)$$

- Configuration 4:

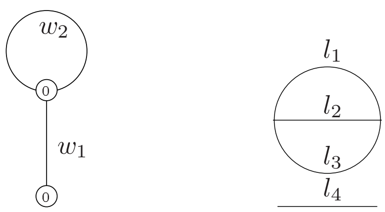


Figure 4.32: Configuration 4

$w_1 = W_1$ is an integer and all remaining parameters are half-integers. There is $1 = \frac{1}{3}\binom{3}{1}$ way to give name to the faces, say

$$\begin{cases} W_1 = 2l_4 = l_1 + l_3 \\ w_2 = l_1 + l_2 = l_2 + l_3 \end{cases}.$$

So there is $\mathbb{1}_{\{2w_2 > W_1\}}$ solution.

- Configuration 5:

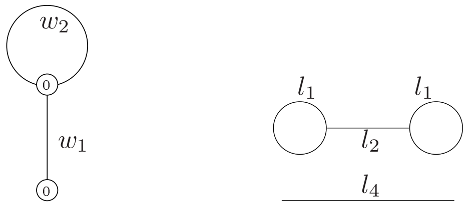


Figure 4.33: Configuration 5

$w_1 = W_1$ is an integer and all remaining parameters are half-integers. There is $1 = \frac{1}{3}\binom{3}{1}$ way to give name to the faces, say

$$\begin{cases} W_1 = 2l_4 = 2(l_2 + l_1) \\ w_2 = l_1 \end{cases}.$$

So there is $\mathbb{1}_{\{2w_2 < W_1\}}$ solution.
Summing the contributions of configurations 4 and 5 we obtain:

$$\sum_{W_1h_1 + w_2h_2 \leq N/2} 2W_1 2w_2 = \sum_{2W_1H_1 + W_2H_2 \leq 2N} 2W_1W_2 \sim 2\frac{1}{4}\frac{(2N)^4}{4!}(\zeta(2))^2 = \frac{5N^4}{6}\zeta(4)$$

(To understand the factor 1/4, take $x_1 = \frac{2W_1H_1}{2N}$ in the proof of Lemma 3.7 of [AEZ2]).

- Total Summing all the contributions, we obtain $\dfrac{15N^4}{4}\zeta(4)$. So by (4.22):

$$\operatorname{Vol} \mathcal{Q}(1^2, -1^2) = 30\zeta(4) = \frac{\pi^4}{3},$$

which coincides with the value found in (4.5).

4.8 Toolbox

Recall that

$$\zeta(2) = \frac{\pi^2}{6}, \quad \zeta(4) = \frac{\pi^4}{90} \quad \text{so } (\zeta(2))^2 = \frac{5}{2}\zeta(4).$$

Lemma 9.

$$\forall m \geq 2, \quad \sum_{k \geq 0} \frac{1}{(2k+1)^m} = \frac{2^m - 1}{2^m}\zeta(m) \tag{4.23}$$

$$\forall m \geq 1, \quad \sum_{i=1}^{N} i^m \underset{N \to \infty}{\sim} \frac{N^{m+1}}{m+1} \tag{4.24}$$

$$\forall m \geq 1, \quad \operatorname{card}\{(l_1, \ldots, l_m) \in \mathbb{N}^m | N = 2l_1 + \cdots + 2l_j + l_{j+1} + \cdots + l_m\} \underset{N \to \infty}{\sim} \frac{N^{m-1}}{2^j (m-1)!} \tag{4.25}$$

We recall the following standard fact (Lemma 3.7 of [AEZ2]):

Lemma 10 (Athreya-Eskin-Zorich)**.**

$$\sum_{\substack{H\cdot W\leq N\\ W\in\mathbb{N}^k, W\in\mathbb{N}^k}} W_1^{a_1+1}\dots W_k^{a_k+1}\sim\frac{N^{a+2k}}{(a+2k)!}\cdot\prod_{i=1}^{k}(a_i+1)\zeta(a_i+2)$$

We will need the following variation of the previous lemma:

Lemma 11.

$$\sum_{W(H_1+2H_2)\leq 2N} W^m \quad\sim\quad \frac{N^{m+1}}{2(m+1)}\left(2^{m+1}\zeta(m)-(2^{m+1}+1)\zeta(m+1)\right) \tag{4.26}$$

Proof.

$$A=\sum_{W(H_1+2H_2)\leq 2N} W^m=\sum_{WH\leq 2N} W^m\,\mathrm{card}\{(H_1,H_2)\in\mathbb{N}^2 \text{ s.t. } H=H_1+2H_2\}$$

Since $2H_2$ is even and goes from 2 to $H-1$ or $H-2$ depending on the parity of H, we have :

$$\mathrm{card}\{(H_1,H_2) \text{ s.t. } H=H_1+2H_2\}=\lfloor\frac{H-1}{2}\rfloor.$$

$$\begin{aligned} A\sim\sum_{WH\leq 2N} W^m\lfloor\frac{H-1}{2}\rfloor \quad &= \quad \sum_{W(2K+1)\leq 2N} W^m K+\sum_{W(2K+2)\leq 2N} W^m K\\ &\sim \quad \sum_{K\geq 1} K\left(\frac{1}{m+1}\left(\frac{2N}{2K+1}\right)^{m+1}+\frac{1}{m+1}\left(\frac{2N}{2K+2}\right)^{m+1}\right)\end{aligned}$$

using (4.24). So

$$A=\frac{N^{m+1}}{m+1}\left(2^{m+1}\underbrace{\sum_{K\geq 0}\frac{K}{(2K+1)^{m+1}}}_{S_1(m)}+\underbrace{\sum_{K\geq 0}\frac{K}{(K+1)^{m+1}}}_{S_2(m)}\right)$$

$$2S_1(m)+\sum_{K\geq 0}\frac{1}{(2K+1)^{m+1}}=\sum_{K\geq 0}\frac{1}{(2K+1)^m}$$

So using (4.23) we obtain:

$$S_1(m)=\frac{1}{2^{m+2}}((2^{m+1}-2)\zeta(m)-(2^{m+1}-1)\zeta(m+1))$$

Similarly,

$$S_2(m)=\zeta(m)-\zeta(m+1),$$

which gives the result.

□

Appendix A

Report on experiments

We give here a small report on some of the experiments that we made to check the formulas given in the previous chapters.

In the following we use the relations of Theorem 8:

$$c_{area}(\mathcal{C}) = \frac{1}{\dim_{\mathbb{C}} \mathcal{Q}(\alpha) - 1} \cdot \frac{4q_1 + q_2}{4} c(\mathcal{C})$$

where q_1 is the number of thin $\hat{\text{h}}$omologous cylinders in the configuration $\mathcal{C}$ in the stratum $\mathcal{Q}(\alpha)$, and q_2 the number of thick cylinders. We denote $\tilde{c}_* = \zeta(2) c_*$ the renormalized constant.

We used Alex Eskin's program which counts configurations in the Abelian case in the following way. Given a stratum $\mathcal{Q}(\alpha)$, for each configuration we compute the corresponding configuration in the double cover. The symbolic to encode configurations used in the output of the program is the saddle connections diagrams. We start to construct an explicit polygon representing a flat surface in $\mathcal{Q}(\alpha)$, that we hope to be generic, using mainly a perturbation of a one-cylinder surface given in [Zo5].

We expect a relation

$$c(\mathcal{C}) = \frac{1}{2} \cdot m(\mathcal{C}) \cdot n(\mathcal{C}, V) \cdot c_{prog}^{startV}(\mathcal{C}),$$

between the constant $c(\mathcal{C})$ responsible for counting configuration $\mathcal{C}$ downstairs, and the output constant $c_{prog}^{startV}(\mathcal{C})$ of the program of Alex Eskin (that counts the corresponding configuration upstairs starting from vertex V), where $\frac{1}{2}$ is the factor responsible for the convention on the area: the area downstairs is one half of the area upstairs. The factor $m(\mathcal{C})$ is responsible for the numbering of the poles downstairs, and the optional factor $n(\mathcal{C}, V)$ is responsible for the normalization of the volume: if only short saddle connections (saddle connections of minimal length) emerge from vertex V, then $n(\mathcal{C}, V) = 1$, if only long saddle connections (saddle connections twice longer) emerge from vertex V, then $n(\mathcal{C}, V) = 4$, if there are p short saddle connections and q long saddle connections emerging from vertex V then $n(\mathcal{C}, V) = \dfrac{4(p+q)}{4p+q}$.

When there are two experiments, the first input is a perturbation of a one-cylinder representative of the strata (given in [Zo5]), and the second is another polygon constructed by hands.

1.1 Hyperelliptic strata

For all these surfaces, we have the expected relation $\tilde{c}_{comp} = \frac{1}{2} \cdot \tilde{c}_{prog}$.

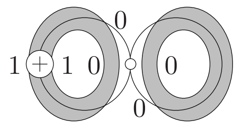
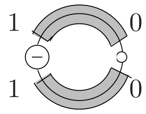
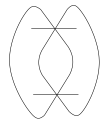
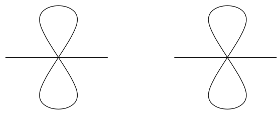
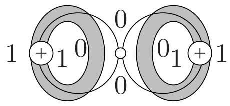
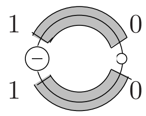
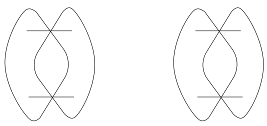
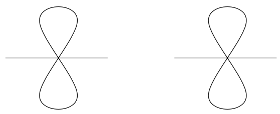
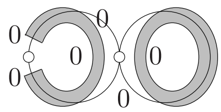
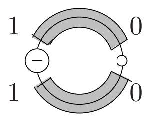
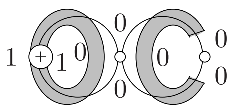
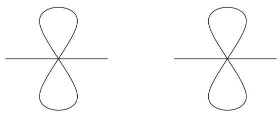
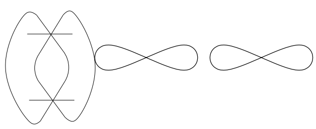
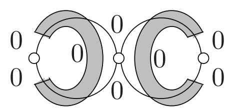
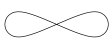
Stratum	Configuration.	Lift of the configuration	$\tilde{c}_{comp}$	$\tilde{c}_{prog}$
$\mathcal{Q}(1^2, -1^2)$	 $\{-1^2\}$ 	 	$\frac{2}{3}$ 1	1.33435 / 1.33251 1.99761 / 2.0024
$\mathcal{Q}(1^4)$	 $\{1^2\}$ 	 	$\frac{5}{9}$ $\frac{5}{4}$	1.11626 2.50411
$\mathcal{Q}(2, -1^2)$			$\frac{1}{2}$	1.00018
$\mathcal{Q}(2, 1^2)$	$\{2\}$  	 	$\frac{10}{9}$ $\frac{5}{9}$	2.20671 / 2.22198 1.11087 / 1.11214
$\mathcal{Q}(2, 2)$		 $\times 4$	$\frac{1}{2}$	0.999642

Figure A.1: Experiments for the hyperelliptic strata

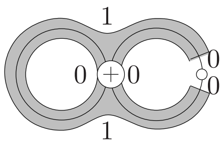
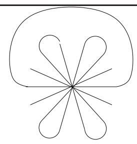
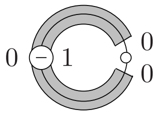
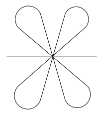
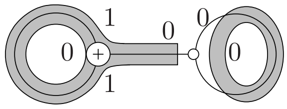
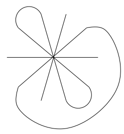
Stratum	Configuration.	Lift of the configuration	$\tilde{c}_{comp}$	$\tilde{c}_{prog}$
$\mathcal{Q}(5,-1)$			$\frac{15}{14}$	1.82339 / 1.8212
$\mathcal{Q}(3,-1^3)$	$\{-1^3\}$ 		$\frac{3}{5}$	1.19857 / 1.18898
			$\frac{2}{5}$	0.603081 / 0.603282

Figure A.2: Experiments for $\mathcal{Q}(5,-1)$ and $\mathcal{Q}(3,-1^3)$

1.2 Strata for which volumes are computed: $\mathcal{Q}(5,-1)$ and $\mathcal{Q}(3,-1^3)$

For the stratum $\mathcal{Q}(5,-1)$, one of the five emerging saddle connections is twice longer as the others, so $n(\mathcal{C},V)=\frac{20}{17}$ and we have $\tilde{c}_{comp}=\frac{1}{2}\cdot\frac{20}{17}\cdot\tilde{c}_{prog}\simeq 1.07$ as expected.

For the stratum $\mathcal{Q}(3,-1^3)$ we have $\tilde{c}_{comp}(\mathcal{C}_1)=\frac{1}{2}\cdot\tilde{c}_{prog}(\mathcal{C}_1)$ as expected and $\tilde{c}_{comp}(\mathcal{C}_2)=3\cdot\frac{1}{2}\cdot\tilde{c}_{prog}(\mathcal{C}_2)$ as expected: the factor $m(\mathcal{C})=3$ is responsible for the numbering of the poles.

1.3 Another example: $\mathcal{Q}(5,1,-1^2)$

For this stratum, we denote $c'_{comp}(\mathcal{C})=c(\mathcal{C})\cdot\frac{\mathrm{Vol}(\mathcal{Q}_1(5,1,-1^2))}{\pi^4}$ since we do not know the value of the volume a priori.

All experiments start from the vertex of valency 6, except for the two first ones. Since all saddle connections emerging to this vertex are minimal, there are no factors responsible for the normalization. Also there is no multiplicity for each configuration, so we expect to find $\tilde{c}_{comp}=\frac{1}{2}\cdot\tilde{c}_{prog}$ for each configuration. The ratios of the constants for the configurations seem to be correspond with the experiments. Since this stratum is non varying ([CM]), we know its Siegel–Veech constant:

$$c_{area}=\frac{97}{42\pi^2}.$$

With this value, we obtain a volume equal to:

$$\mathrm{Vol}\,\mathcal{Q}_1(5,1,-1^2)=\frac{7}{30}\pi^6.$$

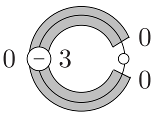
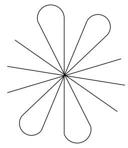
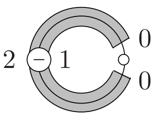
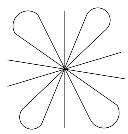
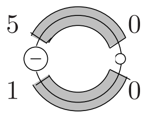
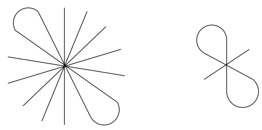
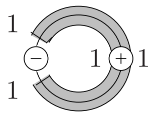
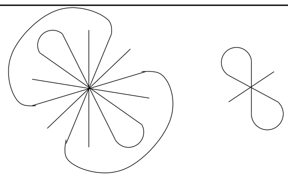
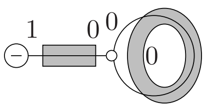
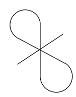
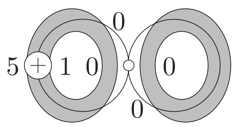
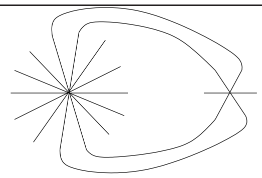
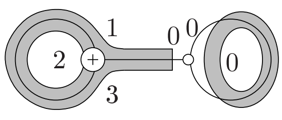
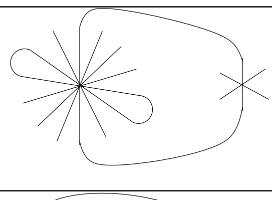
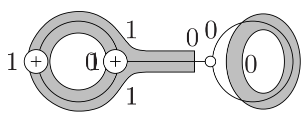
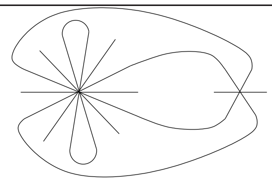
Configuration	Lift of the configuration	c'_{comp}	$\tilde{c}_{prog}$	c_{area}/c
$\{1,-1^2\}$ 		$\frac{1}{2}$	0.677673	$\frac{1}{5}$
$\{1,-1^2\}$ 		$\frac{1}{2}$	0.773203	$\frac{1}{5}$
$\{-1^2\}$ 		$\frac{25}{18}$	2.01102	$\frac{1}{5}$
$\{-1^2\}$ 		$\frac{1}{9}$	0.174377	$\frac{1}{5}$
$\{5\}$ 		$\frac{14}{135}$	0.1311	$\frac{1}{20}$
		$\frac{1}{5}$	0.309696	$\frac{1}{20}$
		$\frac{2}{5}$	0.281827+0.278267	$\frac{1}{20}$
		$\frac{2}{27}$	1.0109628	$\frac{1}{20}$

Figure A.3: Experiments for $\mathcal{Q}(5,1,-1^2)$

Bibliographie

[AEZ1] J.S. Athreya, A. Eskin, A. Zorich, *Right-Angled Billiards and Volumes of Moduli Spaces of Quadratic Differentials on* $\mathbb{C}P^1$, `arXiv:1212.1660` (2012).

[AEZ2] J.S. Athreya, A. Eskin, A. Zorich, *Counting Generalized Jenkins-Strebel Differentials*, Geom. Dedicata, **170** (2014), pp. 195–217.

[Ba1] M. Bainbridge, *Euler Characteristics of Teichmüller Curves in Genus Two*, Geom. Topol., **11** (2007), pp. 1887–2073.

[Ba2] M. Bainbridge, *Billiards in L-shaped tables with barriers*, GAFA, **20:2** (2010), pp. 299–356; **20:5** (2010), 1306.

[BG] M. Bauer, E. Goujard, *Geometry of periodic regions on flat surfaces and associated Siegel-Veech constants* , `arXiv:1405.4748` (2014).

[Bo] C. Boissy, *Configurations of saddle connections of quadratic differentials on* $\mathbb{CP}^1$ *and on hyperelliptic Riemann surfaces*,Comment. Math. Helv., **84:4** (2009), pp. 757–791.

[BM] I. Bouw, M. Möller, *Teichmüller curves, triangle groups, and Lyapunov exponents*, Annals. of Maths., **172:1** (2010), pp. 139–185.

[Ch1] D. Chen, *Square-tiled surfaces and rigid curves on moduli spaces*, Adv. Math., **228:2** (2011), pp. 1135–1162.

[Ch2] D. Chen, *Covers of the projective line and the moduli space of quadratic differentials*, Geom. Dedicata, **163:1** (2013), pp 105–125.

[CkE] J. Chaika and A. Eskin, *Every flat surface is Birkhoff and Osceledets generic in almost every direction*, `arXiv:1305.1104` (2013).

[CM] D. Chen, M. Möller, *Quadratic Differentials in Low Genus: Exceptional and Non-varying*, Annales scientifiques de l'ENS, **47:2** (2014).

[De] P. Deligne, *Un théorème de finitude pour la monodromie*, Discrete groups in geometry and analysis (New Haven, Conn., 1984), Progr. Math., Birkhäuser Boston., **67** pp. 1–19.

[Du] J. Dutka, *The incomplete beta function – a historical profile*, Archive for History of Exact Sciences, **24:1** (1981), pp. 11–29.

[DHL] V. Delecroix, P. Hubert, S. Lelièvre, *Diffusion for the periodic wind-tree model*, `arXiv:1107.1810` (2011).

[EKZ1] A. Eskin, M. Kontsevich, A. Zorich, *Lyapunov spectrum of square-tiled cyclic covers,* J. Mod. Dyn., **5:2** (2011), pp. 319–353.

[EKZ2] A. Eskin, M. Kontsevich, A. Zorich, *Sum of Lyapunov exponents of the Hodge bundle with respect to the Teichmüller geodesic flow*, Publications mathématiques de l'IHÉS, Springer Berlin Heidelberg (2013).

[EMa] A. Eskin, H. Masur, *Asymptotic formulas on flat surfaces*, Ergodic Theory and Dynamical Systems, **21:2** (2001), pp. 443–478.

[EMZ] A. Eskin, H. Masur, A. Zorich, *Moduli Spaces of Abelian Differentials: The Principal Boundary, Counting Problems, and the Siegel–Veech Constants*, Publications de l'IHES, **97:1** (2003), pp. 61–179.

[EMi] A. Eskin, M. Mirzakhani, *Invariant and stationary measures for the SL(2,R) action on Moduli space*, `arXiv:1302.3320` (2013).

[EMM] A. Eskin, M. Mirzakhani, A. Mohammadi, *Isolation, equidistribution, and orbit closures for the SL(2,R) action on Moduli space*, `arXiv:1305.3015` (2013).

[EOk1] A. Eskin, A. Okounkov, *Asymptotics of number of branched coverings of a torus and volumes of moduli spaces of holomorphic differentials*, Inv. Math., **145:1** (2001), pp. 59–104.

[EOk2] A. Eskin, A. Okounkov, *Pillowcases and quasimodular forms*, Algebraic Geometry and Number Theory, Progress in Mathematics, **253** (2006), pp 1–25.

[EOPa] A. Eskin, A. Okounkov, R. Pandharipande, *The theta characteristic of a branched covering*, Adv. Math., **217:3** (2008), pp. 873–888.

[Fi] S. Filip, *Splitting mixed Hodge structures over affine invariant manifolds*, `arXiv:1311.2350` (2013).

[Fo1] G. Forni, *Deviation of ergodic averages for area-preserving flows on surfaces of higher genus,* Ann. of Math. (2), **155:1** (2002), pp.1–103.

[Fo2] G. Forni, *On the Lyapunov exponents of the Kontsevich-Zorich cocycle*, Handbook of dynamical systems. Volume 1B. Editors: B. Hasselblatt and A. Katok. Elsevier, Amsterdam, (2006) pp. 549–580.

[FMZ1] G. Forni, C. Matheus, A. Zorich, *Lyapunov spectrum of invariant subbundles of the Hodge bundle,* `arXiv:1112.0370v3` (2011), to appear in Ergod. Theory Dyn. Syst.

[FMZ2] G. Forni, C. Matheus, and A. Zorich, *Zero Lyapunov exponents of the Hodge bundle.* `arXiv:1201.6075v3` (2012), to appear in Comment. Math. Helv.

[Go1] E. Goujard, *A criterion for being a Teichmüller curve*, Math. Res. Lett., **19:4** (2012), pp. 847–854.

[Go2] E. Goujard, *Siegel–Veech constants and volumes of strata of moduli spaces of quadratic differentials* `arXiv:1405.5899` (2014).

[GJ] E. Gutkin, C. Judge, *Affine mappings of translation surfaces: Geometry and arithmetic*, Duke Math Journal, **103:2** (2000), pp. 191–213.

[Hu] J.H. Hubbard, *Teichmüller theory and applications to geometry, topology, and dynamics. Vol. 1,* Matrix Editions, Ithaca, NY, 2006.

[JV] D. M. Jackson, T. I. Visentin, *An atlas of the smaller maps in orientable and nonorientable surfaces*, CRC Press Series on Discrete Mathematics and its Applications, Chapman & Hall/CRC,Boca Raton, FL (2001).

[Ko1] M. Kontsevich, *Intersection theory on the moduli space of curves and the matrix Airy function*, Commun. Math. Phys., **147** (1992), pp. 1–23.

[Ko2] M. Kontsevich, *Lyapunov exponents and Hodge theory*, in "The mathematical beauty of physics (Saclay, 1996)", Adv. Ser. Math. Phys., **24** pp. 318–332. World Sci. Publ., River Edge, NJ (1997).

[KZ] M. Kontsevich, A. Zorich, *Connected components of the moduli space of Abelian differentials with prescribed singularities*, Invent. Math., **153** (2003), pp. 631–678.

[KtZg] D. Korotkin, P. Zograf, *Tau function and moduli of differentials*, Math. Res. Lett., **18:3** (2011), pp. 447–458.

[La1] E. Lanneau, *Hyperelliptic components of the moduli spaces of quadratic differentials with prescribed singularities*, Comment. Math. Helv.**79** (2004), pp. 471–501.

[La2] E. Lanneau, *Connected Components of the Strata of the Moduli Spaces of Quadratic Differentials* Ann. Sci. Ec. Norm. Sup. **41** (2008), pp. 1–56.

[Li] K. A. Lindsey, *Counting invariant components of hyperelliptic translation surfaces*, `arXiv:1302.3282`.

[Ma1] H. Masur, *Interval exchange transformations and measured foliations*, Ann. of Math., **115** (1982), pp. 169–200.

[Ma2] H. Masur, *Lower Bounds for the number of saddle connections and closed trajectories of a quadratic differential*, in *Holomorphic functions and Moduli*, Vol. I, David Drasin, ed., Springer Verlag, (1988).

[Ma3] H. Masur, *The growth rate of trajectories of a quadratic differential*, Erg. Theory Dyn. Syst., **10** (1990), pp. 151–176.

[Mö06] M. Möller, *Variations of Hodge structures of a Teichmüller curve,* J. Amer. Math. Soc., **19:2** (2006), pp. 327–344 (electronic).

[Mö] M. Möller, *Park City Lectures notes: Teichmüller curves, mainly from the viewpoint of algebraic geometry.*

[MS] H. Masur, J. Smillie, *Quadratic differentials with prescribed singularities and pseudo- Anosov diffeomorphisms*, Comment. Math. Helvetici **68** (1993), pp. 289–307.

[MT] H. Masur, S. Tabachnikov, *Rational billiards and flat structures*, Handbook of dynamical systems, Vol. 1A, North-Holland, Amsterdam, (2002), pp. 1015–1089.

[MZ] H. Masur, A. Zorich, *Multiple Saddle Connections on Flat Surfaces ans the Principal Boundary of the Moduli Spaces of Quadratic Differentials*, Geom. Funct. Anal., **18:3** (2008), pp. 919–987.

[Na] Y. Naveh, *Tight upper bounds on the number of invariant components on translation surfaces*, Israel J. Math., **165** (2008), pp. 211–231.

[Os] V.I. Oseledets, *A multiplicative ergodic theorem. Lyapunov characteristic numbers for dynamical systems.* Trans. Moscow Math. Soc., **19** (1968), pp. 197–231.

[Pe84] C.A.M. Peters, *A criterion for flatness of Hodge bundles over curves and geometric applications,* Math. Ann., **268:1** (1984), pp. 1–19.

[Pe00] C. Peters, *Arakelov-type inequalities for Hodge bundles,* Prépublication de l'Institut Fourier no 511, (2000).

[Sc] W. Schmid, *Variation of Hodge structure: the singularities of the period mapping,* Invent. Math., **22** (1973), pp. 211–319.

[Ve1] W. Veech, *Gauss measures for transformations on the space of interval exchange maps*, Ann. Math., **115:1** (1982), pp. 201–242.

[Ve2] W. Veech, *Moduli spaces of quadratic differentials*, J. D'analyse Math., **55** (1990), pp. 117–171.

[Ve3] W. Veech, *Siegel Measures*, Ann. Math., **148** (1998), pp. 895–944.

[Vo] Ya. Vorobets, *Periodic geodesics of translation surfaces*, in "Algebraic and Topological Dynamics", S. Kolyada, Yu. I. Manin and T. Ward (Editors), Contemporary Math., **385**, Amer. Math. Soc., Providence, RI, (2005), pp. 205–258.

[VZ] E. Viehweg, K. Zuo, *Arakelov inequalities and the uniformization of certain rigid Shimura varieties.* J. Differential Geom., **77** (2007), pp.291–352.

[Wi] A. Wilkinson, *Conservative partially hyperbolic dynamics,* in "Proceedings of the International Congress of Mathematicians. Volume III," pp. 1816–1836, New Delhi, (2010), Hindustan Book Agency.

[Wr1] A. Wright, *Schwarz triangle mappings and Teichmüller curves: the Veech-Ward-Bouw-Möller curves,* `arXiv:1203.2685` (2012).

[Wr2] A. Wright, *The field of definition of affine invariant submanifolds of the moduli space of abelian differentials*, `arXiv:1210.4806` (2012).

[Wr3] A. Wright, *Cylinder deformations in orbit closures of translation surfaces* `arXiv:1302.4108` (2013).

[Zo1] A. Zorich, *Asymptotic Flag of an Orientable Measured Foliation on a Surface*, in "Geometric Study of Foliations", World Scientific Pb. Co., (1994), pp. 479–498.

[Zo2] A.Zorich, *How do the leaves of a closed 1-form wind around a surface*, in "Pseudoperiodic Topology", V. Arnold, M. Kontsevich, A. Zorich (eds.), Translations of the AMS, Ser.2, vol. **197**, AMS, Providence, RI (1999), pp. 135–178.

[Zo3] A. Zorich, *Square tiled surfaces and Teichmüller volumes of the moduli spaces of Abelian differentials*, in collection "Rigidity in Dynamics and Geometry", M.Burger, A.Iozzi (Editors), Springer Verlag, 2002, pp. 459–471.

[Zo4] A. Zorich, *Flat surfaces*, in "Frontiers in Number Theory, Physics and Geometry, vol. 1 : On random matrices, zeta functions and dynamical systems". P. Cartier; B. Julia; P. Moussa; P. Vanhove (Editors), Springer Verlag (2006), pp. 439–586.

[Zo5] A. Zorich, *Explicit Jenkins-Strebel representatives of all strata of Abelian and quadratic differentials*, Journal of Modern Dynamics, **2:1** (2008), pp. 139-185.